# PROJECT DESCRIPTION

## Prototype Software for Boundary Mathematics Curriculum Development

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Although the group theoretic foundations of modern algebra are exceptionally well suited for machines that process strings and streams, they do not support human intuition based on experiences within physical space. We wish to explore what Lee Shulman calls the *pedagogical content knowledge* of mathematics, that particular form of mathematics that is most germane to its learnability, what makes mathematics comprehensible to a student.

Digital technology has expanded the domain of representation of mathematical concepts from typographical strings to spatial forms and to virtual manipulatives. These systems support Bruner's three types of representation for mathematical operations: enactive, symbolic, and iconic [1]. Educators see spatial and physical representations as a possible route to enhance math comprehension [2]:

> "Physical representations serve as tools for mathematical communication, thought, and calculation, allowing personal mathematical ideas to be externalized, shared, and preserved....mathematical ideas are enhanced through multiple representations, which serve not merely as illustrations or pedagogical tricks but form a significant part of the mathematical content and serve as a source of mathematical reasoning." [3]

There is abundant evidence that interactivity can assist students who are having difficulty learning abstract material. Students find it easier to master algebra if it is made concrete through the use of manipulatives [4]. Spatial representations enhance understanding, since expressing ideas spatially allows information to be analyzed more effectively by parallel perceptual processes than by linear cognitive processes [5][6]. Many different ways of making mathematical concepts more concrete have been shown to be effective in learning algebra [7][8].

One direction of growth in tools for teaching mathematics is the use of virtual manipulatives. A virtual manipulative is "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" [9]. Virtual manipulatives provide iconic models that simulate concrete manipulation. Students can construct meaning by using computer input devices to control apparently physical interaction with virtual objects through translation, rotation, flipping and other spatial transformations.

However, diversity of representation is generally shunned within formal mathematics:

> "...despite the obvious importance of visual images in human cognitive activities, visual representation remains a second-class citizen in both the theory and practice of mathematics." [10]

## BOUNDARY MATHEMATICS

We have developed *boundary mathematics*, a family of spatial representations and relations that axiomatize foundational systems, including predicate logic and the arithmetic and algebra of numbers [11]. Boundary mathematics provides alternative models of computation and proof that incorporate parallelism, visual and physical interaction, and simple algebraic transformation based on spatial pattern-matching.

We propose to develop prototype software for the college algebra curriculum that provides an infrastructure for:

- comparison of string-based and spatial axiom systems (comparative axiomatics),
- research into the educational psychology of spatial mathematics,
- development of curricula, and
- comparative evaluation of learning under different axiomatic approaches.

We see these potential benefits:

- An alternative spatial notation may assist the teaching and learning of some concepts by some students, since diagrammatic and three-dimensional formal systems permit visual, manipulative and physical interaction with abstract concepts.
- A comparative axiomatics may enrich our teaching of mathematical concepts.
- Comparison of the errors made in each system can guide research insight into the semantic and syntactic sources of math miscomprehension [12].
- Spatial axioms can more adequately incorporate parallelism into mathematical models and computational algorithms [13].
- Spatial representations may help to bridge the gap between concrete understanding and abstract symbol manipulation, by providing tools that more closely align the abstract formality of mathematics with our concrete experience of real objects.

Boundary mathematics is not a visual representation of conventional mathematical form, it is instead the beginning of a fundamental reconstruction of foundations, using two- and three-dimensional objects and spaces in place of one-dimensional token-strings to represent formal concepts. There is very little published research on boundary math systems. Boundary logic is the most studied, beginning with Peirce [14][15][16][17], and followed by Spencer-Brown [18], Kauffman [19][20][21], Varela [22], Bricken [23][24], and Shoup [25]. Kauffman has made original contributions to boundary numerics [26][27], as have Bricken [28] and James [29][30].

Our short-term objective for this Phase I proposal is to provide interactive animation software that permits calculation and proof via virtual manipulation of spatial forms. The content will be organized in curriculum units that parallel the conventional aspects of the college algebra curriculum. We will develop the software using rapid prototyping techniques, driven by formative evaluation in classrooms. This software is intended to make boundary mathematics systems generally available for detailed exploration and evaluation of their educational, psychological, and mathematical potential.

## Toward the Return of Spatial Intuition into Formal Systems

About one hundred years ago, well prior to the advent of digital display technologies, David Hilbert exerted his considerable influence to complete the exclusion of diagrams and spatial intuition from mathematics and consequently from the mathematics curriculum [31]. The success of the entirely symbolic approach, followed closely by the rise of the use of string-processing techniques in digital computers, has led to the expression of higher mathematics almost exclusively in symbolic string languages.

> "...for two centuries mathematics has had harsh words to say about visual evidence. The French mathematicians around the time of Lagrange got rid of visual arguments in favor of the purely verbal-logical (analytic) arguments that they thought more secure." [32]

> "[A diagram]...is only an heuristic to prompt certain trains of reference:... it is dispensable as a proof-theoretic device....proof is a syntactic object consisting only of sentences arranged in a finite and inspectable array." [33]

What is missing to date are rigorous diagrammatic and spatial systems for logic and for numerics, systems that are inherently interactive and manipulable while at the same time meeting all criteria of symbolic formality. The boundary mathematics systems that we propose to implement meet these criteria. They are based on visual/interactive axioms, and provide what Philip Davis calls visual theorems:

> "...a visual theorem is the graphical or visual output from a computer program - usually one of a family of such outputs - which the eye organizes into a coherent, identifiable whole and which is able to inspire mathematical questions of a traditional nature or which contributes in some way to our understanding or enrichment of some mathematical or real world situation." [34]

At the turn of the 20th century, C.S. Peirce developed a formal spatial system for predicate logic called Existential Graphs [14]. This boundary logic exemplifies the conceptual and representational changes of perspective underlying a comparative axiomatics for logic. Peirce's logic expresses inference by the crossing of a spatial boundary that separates antecedent from consequent. Assertion is confounded with existence, so that False statements are simply erased. Existential Graphs do not have a functional interpretation, since any number of boundaries may be crossed by a single act of spatial inference. A modern algebraic axiomatization of Peirce's spatial logic is presented below [11]. These two equations illustrate both the elegance of the spatial formulation of propositional calculus, and the quite non-standard use of containers and voids to represent logical concepts. The curly braces represent any degree of nesting, including none. The axioms of boundary logic:

```
(A ( )) =
A {A B} = A {B}
```

Since the approach of boundary mathematics is both unfamiliar and unexplored, we next present three spatial mathematical systems that exemplify the potential impact of spatial axiomatization on the arithmetic, algebra, and calculus of quantity. The first is a simple case of base-1 arithmetic of rationals (tally systems) that clearly contrasts differences between string-based and spatial conceptualizations. The second is a spatial algebra that maps addition onto sharing a common space, and multiplication onto touching in space. Finally, we briefly describe elementary "symbolic" differentiation within a spatial calculus of real numbers.

## UNIT-ENSEMBLE ARITHMETIC

Unit-ensemble arithmetic is the arithmetic of strokes and tallies that has been in use for over 30,000 years. A unit is a mark, stroke, notch, pebble, shell, or other discrete singular distinction. Unit marks may be replicated, providing a supply of indistinguishable replicas. Replicate units are intended to be indistinguishable in order to reduce the idea of counting to a foundation of one-to-one correspondence between marks and objects.

Units occupy space in two distinct ways, by sharing a common space and by being partitioned into different spaces (that is, by not sharing a common space). Spatial groupings of units evolved into named quantities in Babylonian, Egyptian, and Roman numerical systems [35][36].

Addition of unit-ensembles consists of placing them into the same space. Multiplication consists of replacing every unit of one ensemble with a replicate of an entire other ensemble. Unit-ensembles differ significantly from sets:

- ✦ there is no empty ensemble
- ✦ the parts (members) of an ensemble are all identical
- ✦ individual units cannot be differentiated by labeling
- ✦ there is no distinction between a single unit and an ensemble consisting of a single unit part, and
- ✦ no unit participates in more than one ensemble.

Over the last several thousand years, humanity has embraced two types of number systems, additive and positional. Both systems support uniform bases and rules that map collections of digit tokens (i.e. names) onto single names. Additive systems, exemplified by unit-ensembles, follow the *Additive Principle* that the representation of a sum is the representations of its parts. Due to the pragmatic inconvenience of reading and computing with additive systems, they have been relegated to early elementary school. To the author's knowledge, there has never been a rigorous axiomatization of unit-ensembles. We present such an axiomatization below, together with a depth-value notation that provides the same benefits for unit-ensembles as does positional notation for digit strings. Thus the example system that follows is both rigorous and efficient, while maintaining the Additive Principle.

## Addition of Unit-Ensembles

A sum in a base-1 additive system is represented directly by its parts. A sum in a group-theoretic system is represented by rules that map ordered pairs onto single objects, together with rules that permit ordering and grouping to be altered. The zero place-holder that supports positional notation is also the group-theoretic additive identity.

Base-1 additive systems have a substantively different axiomatic structure than is presented by the definition of addition in algebra texts. There is no notion of commutativity or of ordered pairs within the concept of ensembles sharing the same space. Addition is achieved by parallel combination of ensembles, using concurrent physical relocation of either or both of the ensembles, or it can be achieved simply by cognitive refocussing of perspective. In both cases addition is the consequence of the removal of spatial partitions.

Teacher training texts recognize the importance of additive systems throughout lower elementary mathematics. However, these texts explain the meaning of addition in terms of symbol manipulation, not in terms of the spatial intuitions of the Additive Principle. The actual structure of additive systems is redefined to fit the group theoretic structure of token-based systems. For example, commutativity of unit-addition is achieved by fiat:

> "We may associate 3+5 with putting a set of 3 members in a dish, and then putting a set of 5 members in a dish to form the union of the sets. We associate 5+3 with putting the 5 set in a dish and then putting in the 3 set." [37]

Unit-ensemble addition, however, does not require or incorporate an external dish nor a temporal ordering of actions. When two unit-ensembles share the same table, children can add them by pushing the piles together concurrently. One ensemble simply does not have temporal or positional precedence over another. The same structure of concurrency applies to parallel addition of three or more ensembles: arity of the operation is simply not a relevant concept.

## Formal Model of Unit-Ensemble Arithmetic

The theory of unit-ensembles is an algebraic theory based on fusion as defined by mereology, and on substitution as defined by the theory of equivalence relations.

| | |
|---|---|
| `• is an ensemble` | (interpreted as +1) |
| `◊ is an ensemble` | (interpreted as –1) |
| `if A and B are ensembles then so is A B` | (interpreted as addition) |
| `no others.` | |

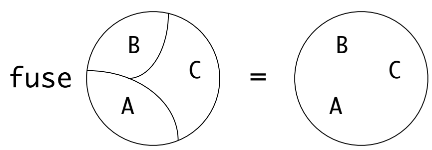
The space shared by ensembles `A` and `B` is *non-metric* and of arbitrary dimension. A consequence is that forms sharing a space are independent of one another. It is the communal space that defines a sum.

The two unit rules that govern the construction of ensembles are:

```
• • ≠ •                 Cardinality
• ◊ =                   Annihilation
```

Annihilation results in *void*, there is no explicit zero. There are no axioms that specify ordering or grouping. The additive inverse is supplanted by a first-class "negative" unit, so that subtraction is a matter of naming rather than an asymmetrical operation.

Addition/subtraction can be understood as the mereological law of fusion. Mereology is the formal study wholes and parts of wholes. Conventional mereology does not embrace an empty object, since the empty whole would have no parts. In general, an ensemble is a mereological whole with indistinguishable parts that do not support overlap (intersection in set theory). Fusion is the removal of partitions between ensembles. It is a *variary, flat* operator that takes an arbitrary number of arguments and that does not support nested application.




```
               fuse(•••|••)  =  •••••
               fuse(A|...|Z) = A...Z
"fuse(A|fuse(B|fuse(C|D)))" = fuse(A|B|C|D) = ABCD
```

Conventional multiplication can be interpreted as substitution of whole ensembles for units. The component units of the ensemble being multiplied are identified by pattern-matching rather than by counting. In pattern substitution, all occurrences of a *for-form* within an *in-form* are replaced in parallel by a *put-form*:

```
substitute(<put-form>, <for-form>, <in-form>)  ==>  <new-in-form>
```

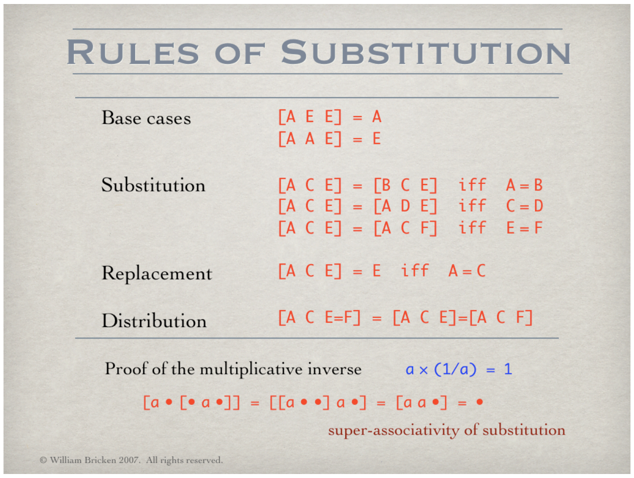
This reads: Concurrently substitute the pattern <put-form> for each literal occurrence of the pattern <for-form> within the pattern environment defined by <in-form>. To introduce a more condensed notation:

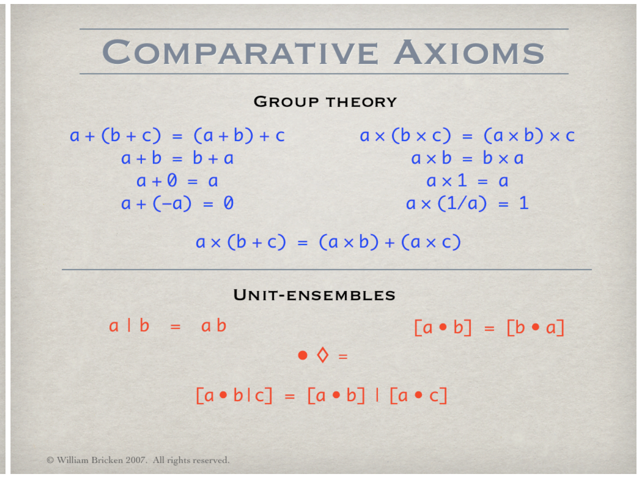
```
substitute(A, C, E)  =notation=  [A C E]
```

Substitution is a property of algebraic equality that follows the rules presented below in the ***Rules of Substitution Figure***. The interpretation of *substitution-as-multiplication* for unit arithmetic is:

```
[A • E]  ==  E*A                [A ◊ E]  ==  –(E*A)
```

Substitution of ensemble `A` for every • (and ◊) in ensemble `E` multiplies the cardinality of `E` by the cardinality of `A`. The sign calculus of multiplication is maintained through adjustment of unit types: substitution ignores type differences, the type of the resulting substitution is toggled (the Δ operator) when the type of the for-form differs from that of the in-form.




The interpretation of substitution-as-multiplication is then defined by two structural constraints: substitution-symmetry between put-forms and in-forms (commutativity), and type-asymmetry between for-forms and in-forms (sign calculus).

```
[A • E] = [E • A]          put/in symmetry
[A ◊ •] = ΔA               for/in type asymmetry
```

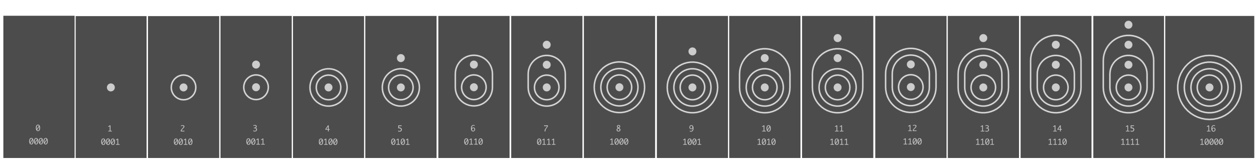
The ***Rules of Substitution Figure*** also includes a simple proof (the multiplicative inverse), to illustrate the fluidity of substitution sequences. Generally, substitution exhibits a type of super-associativity: any form in an odd position can serve as either the put-form or the in-form, any form in an even position can serve as the for-form. That is, substitution-as-multiplication is strongly confluent. Finally, the ***Comparative Axioms Figure*** summarizes an axiomatic basis of unit-ensemble arithmetic and compares it to the conventional axiomatization of whole numbers.

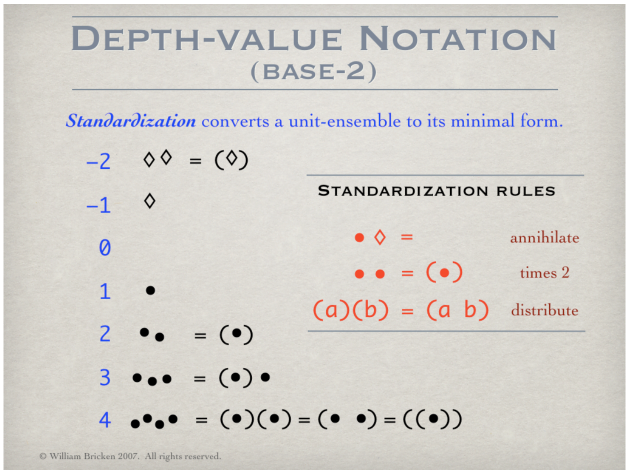
Replacing units by an ensemble defines multiplication; replacing ensembles by units defines division. Division calls for substitution of a single unit for every entire for-form `C` that can be partitioned out of the in-form `E`. The non-commutativity of division is incorporated within the substitution relationship between the for-form and the in-form. The interpretation of *substitution-as-division* for unit arithmetic is:

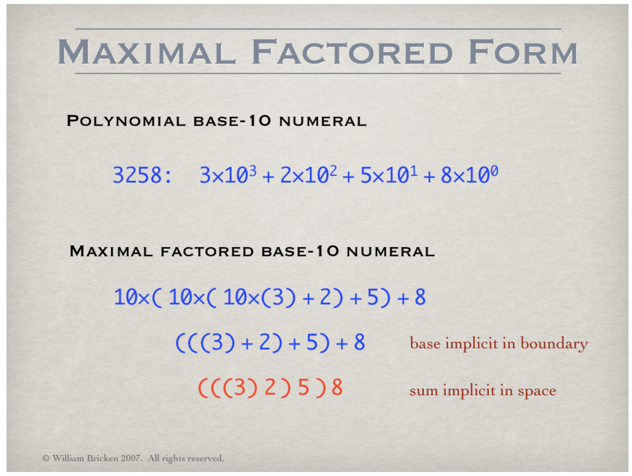
```
[• C E]  ==  E/C          [◊ C E]  ==  -(E/C)
```

## DEPTH-VALUE NOTATION

Unit-ensembles can be rewritten into an efficient depth-value notation by a standardization process that results in a spatial form with minimal structure. The first sixteen base-2 numerals in the resulting container-based boundary number system are presented below. The standardization

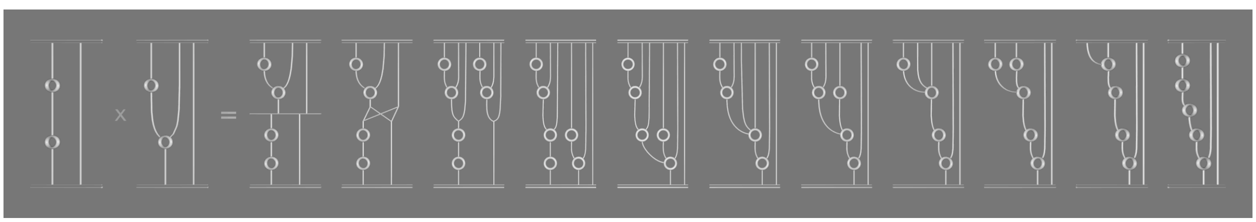




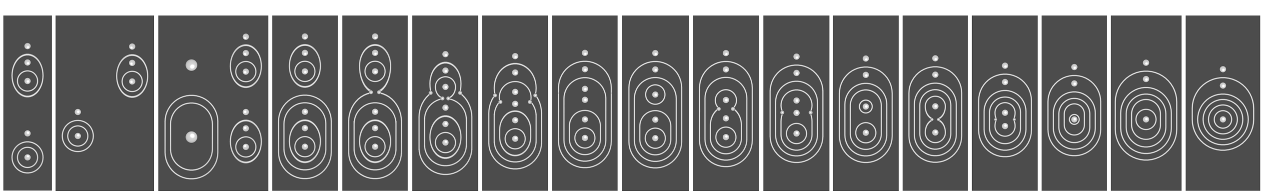

process consists of two rules, *Join Units* (interpreted as increasing the power of the base) and *Join Boundaries* (interpreted as distribution). For clearer presentation, the rules in the ***Depth-value Notation Figure*** are in base-2 rather than base-10. They are recorded in a textual notation that permits recording boundary numbers on typographical lines. The ***Maximal Factored Form Figure*** illustrates another way to interpret the notation of boundary numbers. Conventional numerals are in polynomial form, boundary numerals are in maximally factored form.

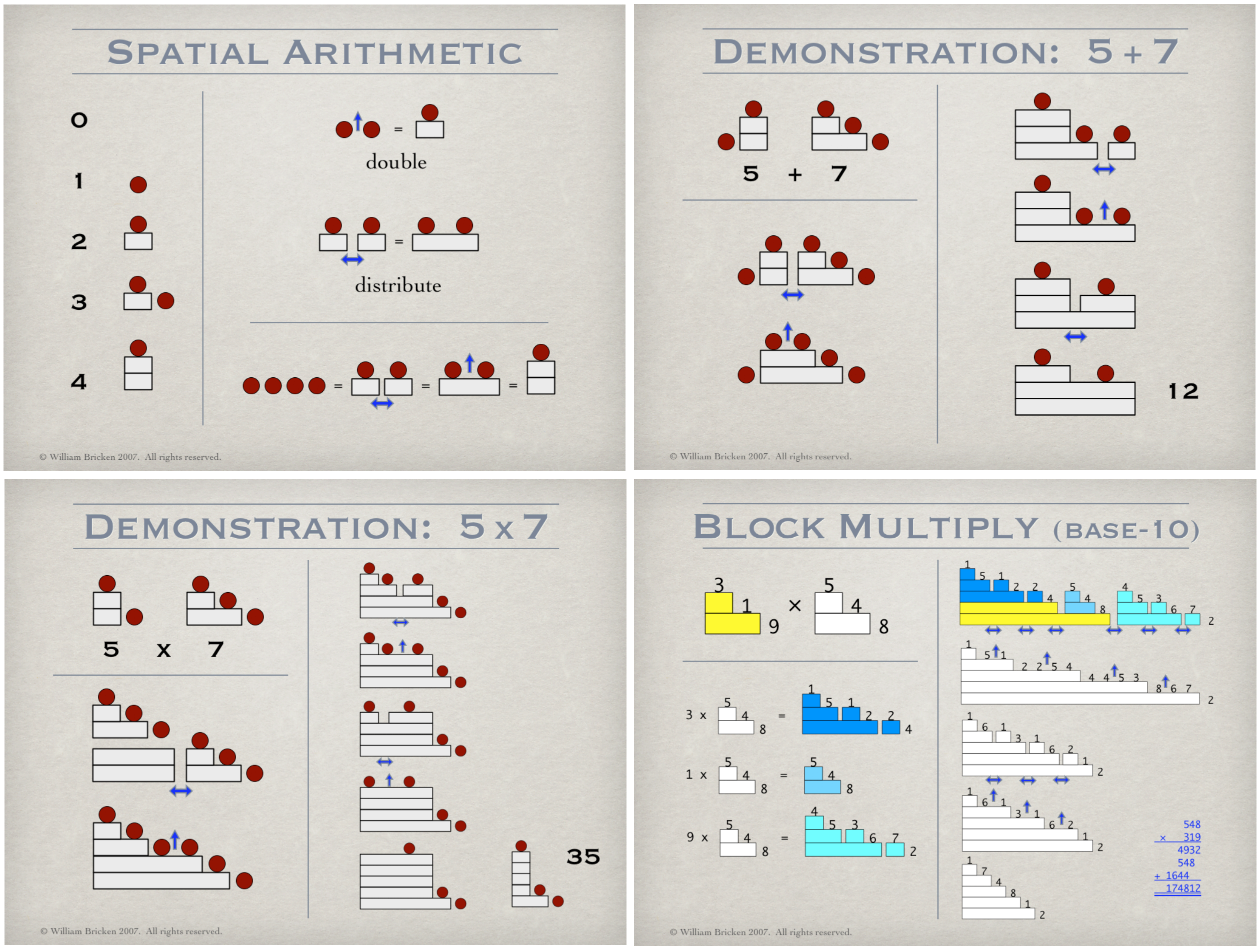
In contrast to the typographical representation that uses delimiters as implicit spatial boundaries, the animation sequence below shows 5x7 in a graph notation obtained by extruding the textual delimiters downward. The unit is represented by a bar at the bottom of the graph.



And for comparison, the animation sequence at the bottom of the page also shows 5x7, this time in the container-based notation. The fourth frame, after the form of 7 is substituted into the form of 5, is equal to 35. The following frames show the dynamic standardization process for containers.

The ***Spatial Arithmetic Figure*** on the next page shows boundary numerals represented in a fourth spatial notation, as stacks of blocks. Although this display is two-dimensional on the page, it is potentially three-dimensional in physical space, and thus can be physically manipulated. The standardization rules for depth-value notation become the physical actions of constructing stacks (doubling in base-2) and pushing stacks together (distribution).



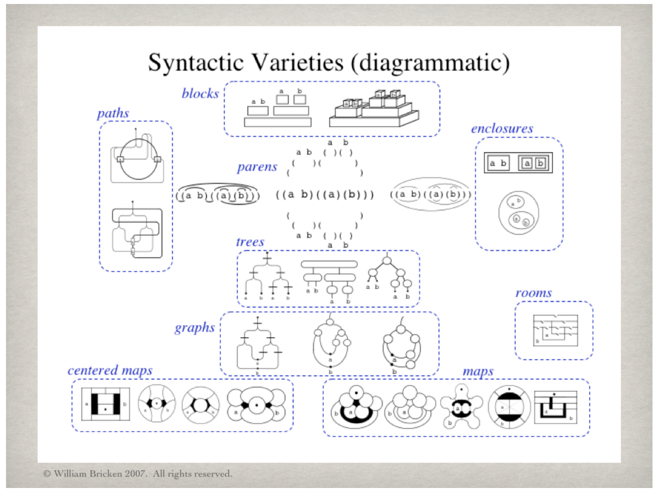



- The ***Demonstration: 5+7 Figure*** shows adding followed by standardization. Addition itself requires no physical effort.
- The ***Demonstration: 5x7 Figure*** shows multiplication followed by standardization. Multiplication itself substitutes a separate replica of 7 for each of the two units of 5.
- The ***Block Multiply (base-10) Figure*** shows multiplication of 319 by 548.

In base-10 spatial multiplication, each digit `M` in 319 is replaced by `M` replicas of 548. Recursively, `M` replicas of 548 are generated by replacing each digit `N` in 548 by the base-10 form of `MxN`. Thus, for example, 3x5 replaces the 5 digit in 548 by the block representation of 15, which is written as `(1)5` using textual delimiters. Naturally, in base-10, the conventional digit addition and multiplication facts (the plus and times tables) are still required to convert between symbolic numerals. Due to the non-existence of an explicit zero, in base-2 there is only one digit fact for addition, in hybrid notation: `1+1 = (1)`. There are no digit facts for multiplication.

## A DIVERSITY OF SPATIAL NOTATIONS

Spatial forms support a diversity of notations, each derived from the others via spatial transformation [24]. These notations include the textual delimiters, containers, graphs, and stacks of blocks as shown above, as well as varieties of other one-, two- and three-dimensional forms such as trees, maps, rooms, and paths. The ***Syntactic Varieties (diagrammatic) Figure*** below presents some of these spatial notations, together with the transformation paths between them.

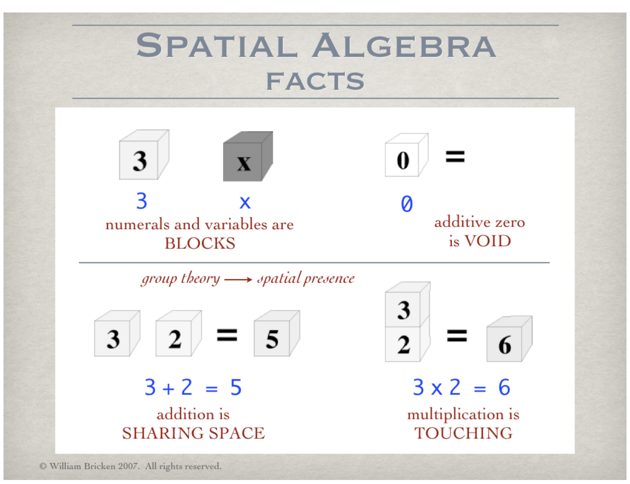


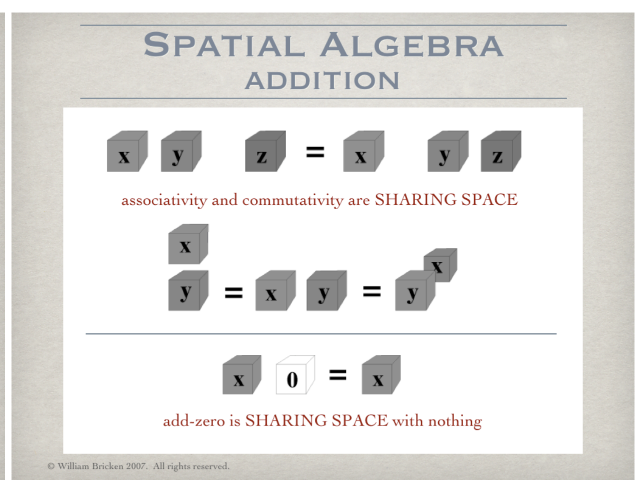

## SPATIAL ALGEBRA

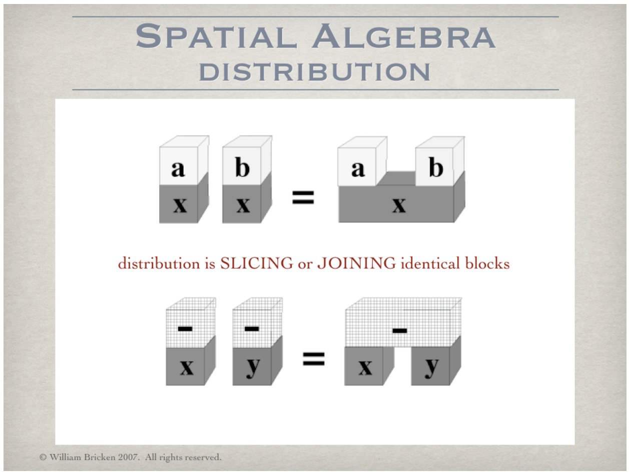
Bricken [28] and Winn and Bricken [38] map abstract algebraic concepts onto properties of space, to construct a different type of boundary mathematics. Addition is represented as sharing the same space without touching, while multiplication is represented by stacks of algebraic objects in physical contact. The ***Spatial Algebra Facts Figure*** illustrates instances of objects and operations within this spatial algebra. The ***Spatial Algebra Addition Figure*** shows that conventional additive concepts of associativity, commutativity, and zero become (irrelevant and misleading) metric distinctions in a non-metric shared space. The ***Spatial Algebra Distribution Figure*** on the following page shows that conventional symbolic distribution is a form of contextual replication achieved by slicing and joining a named block. Finally, the ***Distribution in Depth Figure*** illustrates that the apparently complex nesting of factored symbolic forms can be converted into the equivalent polynomial form by a single parallel act of slicing stacks of spatial objects.

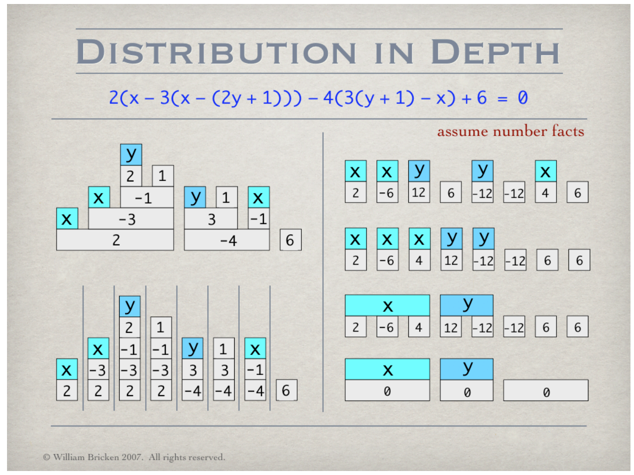
## JAMES CALCULUS

In his thesis under W. Bricken, Jeff James [30] uses three types of spatial containers/boundaries to represent all varieties of numbers (integers, rationals, irrationals, imaginaries). Several unique numerical concepts arise from this approach. We present examples of the generalized inverse, which unifies subtraction, division, roots, and logarithms into a single concept, differentiated by



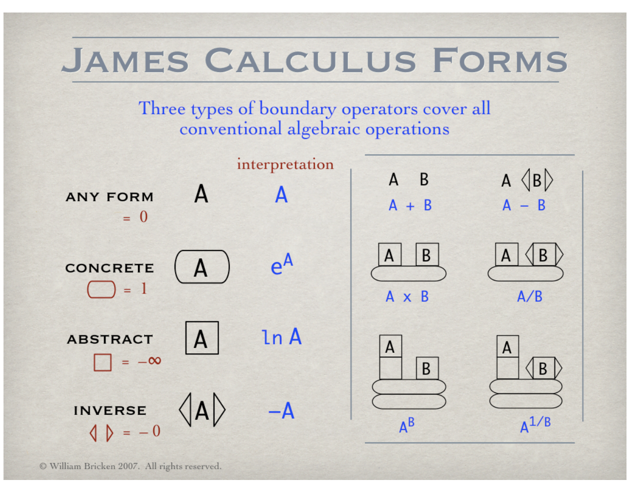


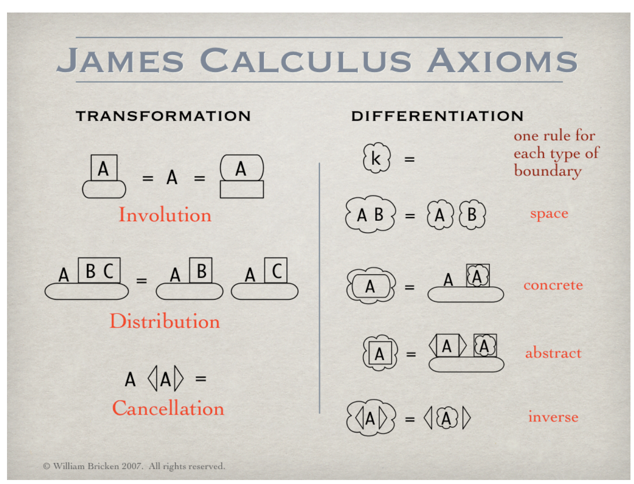



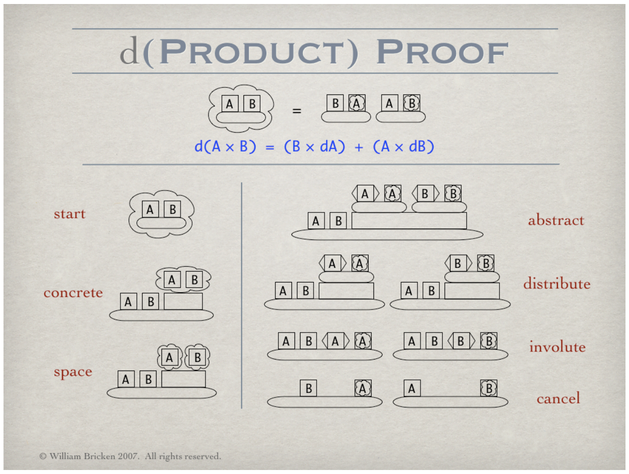

spatial nesting rather than by algebraic operation. The ***James Calculus Forms Figure*** on the net page shows these boundaries, together with their conventional interpretation.

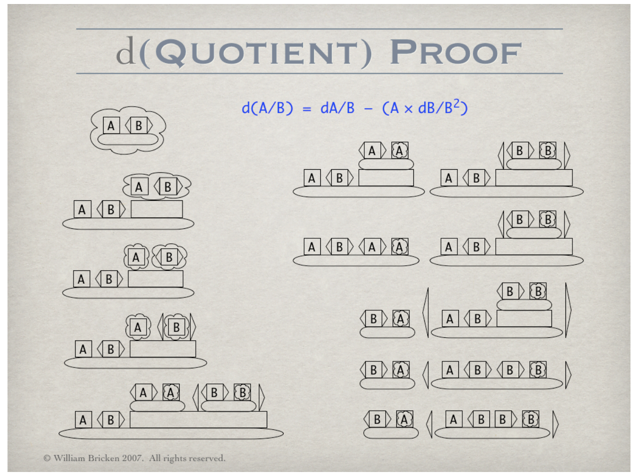
The round container, `( )`, raises `e` to the power of its contents. When it is empty, the value of the boundary is `e^0`, which can be interpreted as the object one. The square container, `[ ]`, takes the logarithm of its contents. An empty square container has the value of `ln 0`, which is negative infinity. The angle container, `< >`, converts its contents to a generalized inverse. An empty angle container has the value `-0`. Thus, the set of empty containers ground arithmetic in its three fundamental values: 0, 1, and infinity.

The ***James Calculus Forms Figure*** also shows the block representation of each of the conventional arithmetic operators (sum, difference, product, quotient, power and root). Note that the angle container represents all three inverses via its position in a stack. Care should be taken in reading these forms, since the notation in the figures is hybrid; stacking and containment mean the same thing. Naturally, the boundary containment and stacking notations can be converted into any of the other spatial forms of representation presented in the ***Syntactic Varieties (diagrammatic) Figure***. We emphasize that these notational choices are at this time rather arbitrary, pending research into their relative effectiveness.








The ***James Calculus Axioms Figure*** shows the three axioms that govern computation. The figure also includes the axioms of James calculus differentiation, one axiom for each form of containment, including void and sharing space. The derivative dx is represented by containing x in a *cloud*. All James forms have a direct interpretation in standard notations, even during transformation steps, however the routes that James forms take to achieve computation are generally very unusual. Without detailed discussion, we present spatial proofs of the product and quotient rules for symbolic differentiation in the ***d(Product) Proof Figure*** and the ***d(Quotient) Proof Figure***. The quotient rule proof incorporates two theorems about the inverse boundary:

<A><B> = <A B>          (A [<B>]) = <(A [B])>

## PLAN OF WORK

The software will provide open access to the axiom systems presented in this proposal, and will be designed to facilitate educational research and comparative axiomatics. All systems will be prototyped within the Mathematica 6.0 computational and programming environment. The developed prototype will support stepwise solution of both computational problems and spatial proofs, and will output performance statistics and raw data for any input problem. All data will be collected in the Mathematica environment, so that statistical analysis tools would be immediately available. The first iteration should take about two or three months. This would leave 6-9 months to develop and refine the display and interaction of the tool. Development will include three cycles of rapid prototype implementation, each with maximal exposure to student feedback. We are currently installing a Math Lab (capacity ~30) which will support lab-oriented math classes at LWTC, and incidentally support formative evaluation of the boundary math software. For Year I, we expect to constrain the usage of the developmental software to students within the multimedia and animation programs, and to volunteer students from math classes.

Debugging can be concluded during Autumn of Year II, leaving two quarters to develop structured curricula materials. By the end of Year I, we should be able to present the software prototype at relevant conferences. By Spring of Year II, we should be ready to distribute fully documented prototype systems. We will disseminate documentation and other information on the availability and utility of the tools during the last quarter of the funded project.

### Development Schedule

| | | |
|---|---|---|
| Year I | Autumn: | Iteration 1: rapid prototyping of engines |
| | Winter: | display design |
| | Spring: | display/engine initial integration |
| | Summer: | Iteration 2: rebuild system, write full documentation |
| Year II | Autumn: | test and refine Iteration 2 |
| | Winter: | Iteration 3: classroom exposure |
| | Spring: | classroom testing, interaction refinement |
| | Summer: | package software tool, final reporting |

### Software Architecture

System capabilities:
- interactive spatial display with animation of transformations
- conventional and boundary input/output languages
- convenient control over substitution-rules and transformation steps
- interpretation at all transformation steps
- lemma stashing

User and interface functions:
- select input and output languages
- allocate engine processors (parallelism)
- select representational forms (BM interaction "languages")
  - unit-ensembles, enclosures, blocks, graphs, strings
- input any valid expression to construct the selected form
- apply any valid transform as an interactive animation
- construct pattern lemmas (theorems)
- automated solve, with control over display content and timing.

Support for educational research:
- structured lower division curriculum for each domain
- cross-linking to traditional curricula

## QUALIFICATIONS OF THE PRINCIPAL INVESTIGATOR

William Bricken has a Ph.D. in Mathematical Methods of Educational Research from the Stanford School of Education, with coursework focused on Artificial Intelligence and Statistical Analysis. Prior to that he was cofounder and Principal of one of the first innovative primary schools in Australia, and an Assistant Professor of Social Psychology and Education at Monash Teacher's College, Larnook. His dissertation empirically validated the unique nature of errors made by students learning algebra. Using a range of experimental techniques (multivariate experiment, exploratory factor analysis, protocol analysis, clinical case study, historical review, and direct remediation), he demonstrated that symbolic errors made by novices are neither random nor predictable, rather they are context sensitive, situated, and unique [12].

Dr. Bricken has designed and implemented leading-edge world-class prototype software for intelligent interface, statistical analysis, distributed logic, high-performance inference engines, virtual reality, semiconductor design, optimization, partitioning and layout. This work was conducted primarily in commercial research labs, including seven years at Paul Allen's Interval Research Corporation in Palo Alto, California. In 1988 he founded and was the Director of the Autodesk Research Lab. He co-founded and was the Principal Scientist of the Human Interface Research Lab at the University of Washington, Seattle from 1990 to 1993, pioneering research and development of immersive virtual reality software for construction, maintenance, and interaction in arbitrary virtual environments [39][40]. He has served as a Research Associate Professor of Education at UW, and worked for five years as an Assistant Professor of Software Engineering and Computer Science in the Seattle University Masters Program in Software Engineering. He is currently on the Mathematics faculty at Lake Washington Technical College.

With the help of several others, including Dr. Louis Kauffman (consultant for this project), Dr. Bricken has developed, implemented, and applied most of the known algorithmic techniques of boundary mathematics. Between 1984-1989, he worked on boundary math implementations for advanced logic processing applications. From 1993-2000, he developed applications for predicate calculus, SAT-solving, combinatorial logic optimization, and semiconductor layout optimization. Between 2001-2006, he developed applications for timing and area optimization of netlists, and refined earlier work on the boundary arithmetic and algebra of numbers. Dr. Bricken has not received NSF funding support over the last five years.

## MATHEMATICS AT LAKE WASHINGTON TECHNICAL COLLEGE

The LWTC student body is extremely diverse, providing a unique environment within which to conduct studies of mathematics education. A primary advantage is that LWTC students are preparing directly for entering the job market rather than seeking a general academic education. Many are enrolled in ESL courses. All technical courses at LWTC offer practical, job-related learning experiences that are needed and in demand by industry. Most students enroll in courses that operate on an open-entry, open-exit system. The LWTC Mathematics Department services concrete and abstract math needs for all technical departments of the college, including information technology, manufacturing and transportation technologies, medical and dental technicians, accounting and business services, and biological sciences. LWTC also administers the Lake Washington Technical Academy, a junior/senior year high school with an enrollment of 450 students. The Academy is housed on the LWTC main campus, and provides high school students with the opportunity to earn concurrently a high school diploma, a vocational certificate, and a two-year college degree.

## ADDITIONAL SUPPORT FOR THE PROPOSAL

Wolfram Research has agreed to donate three licenses for Mathematica 6.0 in support of this project. Project software and tools will be disseminated through the Wolfram Research MathWorld and Mathematica 6.0 Demonstration Project web-sites, as well as through the project web-site, conference presentations, and conventional channels of academic publication.

## LONG TERM PROSPECTS

We have taken care to describe three systems of boundary numerics, showing applicability for arithmetic, algebra, and calculus. The uniqueness and simplicity of these spatial math systems suggest that investment in a preliminary examination for math education is worthwhile.

This Phase I proposal does not include a summative evaluation of the pedagogical effectiveness of boundary mathematics. This non-standard approach is intended to finesse the profound issues associated with the very idea that algebra can be constructed without positional notation, without group theory, without set theory, and without Peano-like axioms based in recursive functions. The soundness and completeness of the systems in this proposal can be established by simple mappings to conventional systems. We are not proposing mathematical research, since from experience we have found that such research would distract from the essentially pedagogical goals of the proposal. Our immediate objective is simply to provide tools that facilitate a thorough evaluation of boundary numerics for mathematics education. The intermediate-range objective is to use these tools for structured study, experimentation, and educational research.

Speculation on potential long-term impact is justified by the innovative characteristics of the proposed work. In particular, we will be examining a new axiomatic approach to the foundations of mathematics. Rigorous formal mathematical systems that are also based rigorously on innate human spatial capabilities may provide a grounding for mathematics education reform. Of course, such sweeping change must address the inertia of existing math education practices. There is no apparently easy interface between typographical and digital tools. The suppression of Peirce's Existential Graphs is ample evidence that positive results alone do not change established notational habits. We share Kempe's goal: "...to separate the necessary matter of exact or mathematical thought from the accidental clothing -- geometrical, algebraical, logical, etc." [41]. Thus the long-term vision, of which this proposal is literally a first step, is to integrate different representations under singular abstractions. During the course of this very long process, we hope that the benefits of spatial mathematics can be demonstrated in terms of improved mathematics pedagogy, a greater variety of tools for communication of mathematical ideas, better service in the education of the digital generation including concrete, visual, tactile and experiential learners and those who are disadvantaged by a purely symbolic presentation of content, and a growth of mathematical open-mindedness through the study of comparative axiomatics.