THE DEVELOPMENT AND ASSESSMENT OF A VIRTUAL WORLD FOR TEACHING ELEMENTARY ALGEBRA

William Winn, University of Washington College of Education
William Bricken
December 1991

Project Summary

The two objectives of this project are to improve students' understanding of basic algebra and to demonstrate the effectiveness of Virtual Reality for attaining this end. We have developed a Spatial Algebra in which variables and constants are represented by programmable objects and algebraic operations are performed by moving the objects in space. Students enter the spatial algebra virtual world by donning a helmet that places them in a three-dimensional environment, and a glove that allows them to manipulate virtual objects in natural ways. Interacting with the objects in the world, the students build knowledge of algebra inductively. Pedagogical strategies are seamlessly woven into the virtual environment by varying the ways in which the laws of algebra are enforced. Transformations can also be selectively automated, allowing students to work with a few operations at a time. Data on the performance of tasks in the virtual world, the extent to which knowledge acquired in the virtual world transfers to other tasks, "think aloud" protocols obtained from students in the virtual world, and estimates of the feasibility of implementing Virtual Reality in schools allow an assessment of the effectiveness of the approach for teaching algebra and, potentially, other subjects.

An Introduction to Virtual Reality

Virtual Reality refers to a new conception of computing that fundamentally redefines the way humans and computers interact. It includes a body of techniques that link natural human behavior to environments created by the computer. The participant in VR interacts with hardware which senses natural behavior, such as pointing, looking and moving around. The virtual world that the computer creates responds to these behaviors in "real time", creating the illusion that the participant is part of the environment rather than interacting with it through an interface.

A participant in a virtual environment wears a helmet that contains earphones for three dimensional sound, "eyephones" for the presentation of stereoscopic visual images, and various other behavior transducing devices which map the participant's physical actions onto virtual events. Computer software links behavior and display in real-time, enhancing the participant's perceived presence in the virtual environment.

Participants interact with the virtual world in several ways. Physical movement can be mapped onto movement of the participant's virtual body. Thus, by walking forwards or backwards, the participant moves forwards or backwards in the virtual world. Hand movement and gestures can be tracked by a "data glove" which controls a virtual hand for grasping virtual objects. A spatial joystick can be used for movement and orientation in three dimensional space. Position trackers can locate in space any part of the body they are attached to. In general, any bodily activity can be mapped onto a corresponding activity in the virtual world.

The best way to think about the experience of VR is to look around the physical reality each of us inhabits. When we turn our head, the world holds still while we redirect our attention in the new direction. VR has the same inclusive quality. In physical reality, we perceive objects. VR has objects also, but they do not necessarily exhibit mass. VR objects are programmable, their properties can be arbitrarily changed. VR is a domain which is made to seem real even though it is entirely imaginary.

Problem

The poor performance of many American students in Mathematics is well documented and acknowledged. Generally, many aspects of Mathematics are too complex and too abstract for students, especially less able ones, to understand. More specifically, the symbol system of Mathematics appears arbitrary and has no easily understood relationship to things which students find familiar. When students fail to comprehend the underlying symbolism of algebra, they do not comprehend algebraic processes like factoring or solving equations, and they do not transfer what they learn to practical applications in the real world.

We believe that VR can overcome the problems of algebraic representation, and of symbolic representation in general. In VR students can learn mathematics through direct experience with virtual objects in the virtual world. Rather than reducing experience to algebraic tokens and then having to interpret the result of symbol manipulation, as is the case when students use traditional methods to factor expressions or solve equations, in VR students can touch and sense environmental objects which embody the rules of mathematics directly. The traditional curriculum in algebra concentrates on the techniques of token manipulation, techniques which have no relevance to experience or functionality in the physical world. In VR, students can concentrate on concepts and meaning rather than on syntax and representation. For example, they can experiment with manipulating virtual objects that represent variables, constants and operators that have been programmed to obey particular laws of Mathematics, thus acquiring experience with Mathematical principles that is much more concrete and direct than that acquired by working problems on paper. Put another way, pedagogy can be seamlessly woven into the mathematical rules that govern the virtual world, facilitating knowledge construction and transfer.

Our project is confined to first year algebra, since the greatest gains for students in understanding algebra (and mathematical abstraction in general) can be achieved at this level. Algebra is difficult for many students to master although the rules of algebra are clear and manageable in number. We propose a virtual algebra world as a preliminary test of the potential benefits of VR for mathematics education. This is the initial step of a larger program to develop an experiential mathematics curriculum for high school, based in VR technology. The techniques of spatial algebra can, in general, be applied to any mathematical system.

Appendix A describes an approach to the spatial representation of algebra that maintains the formal structure of algebra while providing direct interaction with its abstractions. We expect that this approach will help students to overcome the most prevalent errors in first-year algebra (precedence and distribution, negative numbers, and fractions).

The project's objectives are therefore as follows:

1. To provide "proof of concept" that VR can lead to marked improvements in students' understanding of abstract mathematical content, exemplified through elementary algebra.

2. To show that VR is particularly successful with students who typically have problems learning abstract mathematical content.

3. To develop a virtual world that embodies the rules of algebra in a way that lets them also function as pedagogical strategies. This requires that we:

a. Develop a representational approach that allows us to embody the objects and rules of algebra in a virtual world. Spatial algebra has been developed for this purpose (Appendix A).

b. Develop pedagogical strategies that will lead to comprehension of algebraic concepts as a result of a student's experience in the virtual algebra world, and embed these into the rules that govern the world.

c. Identify appropriate methods for assessing student comprehension of algebraic concepts and their transfer from the virtual to the real world.

4. To conduct the project in close collaboration with high school mathematics teachers and students.

5. To initialize an experiential mathematics research program.

Relations to work in progress and long-term goals

The objectives of this project arise directly from the investigators' ongoing research programs. Two lines of inquiry are involved. At the Human Interface Technology Laboratory at the University of Washington, innovative research and development are leading to an understanding and realization of VR in information systems, education, and training. The motivation behind this research is to develop effective strategies for the use of VR. It is our belief that an understanding of the way participants behave and learn in virtual worlds is the key to the success of this research agenda. The long-term goal of this work is therefore to develop strategies for using VR in schools on as large a scale as is necessary to help students who have difficulty learning abstract material from conventional instruction. If this project is successful, the next step will be to put VR directly into classrooms.

The second line of inquiry involves work in the Educational Technology Program at the University of Washington. Generally, this work has been directed at identifying and developing instructional theory and strategies relevant to learning via technology. Here, too, the emphasis is on learning processes and pedagogy. The long-term goal of this research is to identify learning problems that are best addressed through technology. This requires the exploration of the symbol systems of various technologies and determining how they affect cognitive processes. More recently, the emphasis of this research has shifted to an examination of how students construct knowledge for themselves and away from the development of instructional prescriptions. The VR project allows testing of a number of hypotheses arising from this research. For example, it allows very direct observation of students constructing knowledge, and thus the examination of strategies use to solve problems, and integrate knowledge into what they already know. It is particularly timely because it requires the development of a completely new set of strategies that rely on perceptual as well as cognitive learning. The VR project promises the consolidation of a lot of theory development and basic research.

The Virtual Reality project is designed to address a significant problem in learning Mathematics, elementary algebra, as an instance of a more generic problem, students' difficulty in developing sound conceptual models of complex, abstract domains. This brief review of literature examines the nature of the difficulty many students encounter as they begin Algebra. Since we propose to exploit the perceptual and constructivist characteristics of VR to overcome these difficulties, we also describe how our project fits in with current thinking about perceptual learning and constructivist applications of technology. We conclude with an explanation of why we expect VR to succeed where other technologies and instructional methods have failed.

1. Learning Algebra

The difficulties children have when they begin to learn algebra are well documented (Bricken, 1987; O'Shea, 1986; Zehavi & Bruckheimer, 1984; Gerace & Mestre, 1982; Sleeman, 1984). Frequently, these difficulties arise from the novelty and abstruse nature of Algebra's symbol system. Thwaites (1982) found that students are often baffled by algebra's non-visual nature, its apparent arbitrariness, its complexity, and how problems are expressed using its symbols. These difficulties arise from students' inability to grasp what algebraic symbols stand for. For instance, students often do not understand what variables are, how letters are used to represent them, or how equality can be used (Rosnick, 1981; Kaput 1978; Bernard & Bright, 1982). A substantial difficulty is that algebraic symbolism itself is ambiguous and context dependent.

If students fail to understand algebraic representation, then the only way they can solve algebra problems is by the rote application of procedures they have memorized. This memorization is brittle, often both over and under generalized, and elaborated by motivations independent of the content of Algebra (Bricken, 1987). Gerace & Mestre (1982) reported that the students they interviewed did not use proper algebraic techniques to solve problems and treated algebra as a rule-based rather than as a concept-based discipline. Unfortunately, the rule-based approach to Algebra is the one that is often taught, even though this does not promote the development of good conceptual models of algebra (Thwaites, 1982; Bright, 1981; Bernard & Bright, 1982; Greeno, 1985).

Since the symbol system of algebra is a major stumbling block to the development of conceptual models, it is not surprising that a number of attempts to help beginning students overcome their difficulties have focused on helping them understand the algebraic way of representing concepts and relationships. (Textbooks do not have a good record of doing this. Rosnick [1980] found that none of the 41 texts he reviewed adequately conveyed the concept of continuous variability without becoming pedantic and cumbersome.) Many of these attempts have started with the assumption that students find it easier to master algebra if it is made concrete through the use of manipulables, an assumption supported by many experimental studies (see the meta-analysis of this research by Sowell, 1989). The symbols of algebra are typically reified either in physical objects or in computer representations or simulations of problems and problem- solving. Thus in the area of equation solving, we find studies of the effectiveness of using pan balances (Austin & Vollrath, 1989), spreadsheets (Watkins & Taylor, 1989), computer-based graphing (Waits & Demana, 1989), and a variety of other strategies (Shumway, 1989). Usually, students can learn rudimentary aspects of algebra from these techniques. It appears therefore that making algebra more concrete by having students manipulate objects or computer-created symbols can be effective, even as early as elementary school (Berman & Friederwitzer, 1989).

The strategies for teaching algebra by reifying it in concrete objects are all to some degree analogical (e.g., balancing an equation is like balancing the pans of a scale). However, analogies in complex systems like algebra are oversimplifications that lead to the development and entrenchment of misconceptions (Spiro, Feltovich, Coulson & Anderson, 1989).

Of course, the need for students to develop more generic, abstract and powerful conceptual models has not been ignored in algebra instruction. For example, Connell & Ravlin (1988) have proposed a microcomputer-based instructional model for teaching linear equations that uses four types of problems to encourage just this kind of development. Students begin by using manipulables. Then they learn to represent the manipulables in sketches. The next step is to internalize the sketches as mental images. Only then is it possible, through further abstraction, to arrive at a truly algebraic conceptual model.

As a second, rather different, example, Wagner (1981) studied students' conservation of algebraic relations under different forms of variable notation. She argued that a key to success in algebra is for students to learn that the meaning of the relationship in an equation does not change when the way the variable is represented changes. Her sample included non-conservers, conservers, and students in a state of transition between the two. The results of her study allowed her to conclude that less than half of her subjects were able to conserve the relation when the variable changed, and that the ability to do so varied by age, gender, and background in Mathematics.

Our purpose of this project, therefore, is to verify that VR can enable beginning students to learn the rudiments of algebra, and its symbol system, in a way that does not interfere with more advanced learning; and that it can enable the development of more advanced and useful conceptual models of algebra. Our confidence that VR can achieve both of these goals is based on research reviewed in the next section.

We believe that there are two main reasons why VR can improve the understanding of algebra and of other abstract domains. First, VR can express algebraic concepts and rules in forms with which the student can interact directly and concretely, thus allowing the student to use perceptual processes in addition to cognitive processes to learn the material. Second, interaction with the virtual world allows students to construct an understanding of algebra for themselves rather than simply absorb what a teacher tells them. This, too, has been shown to lead to better understanding.

Virtual Reality improves upon screen-based diagrammatic display by including the participant in an environment. Inclusive environments have been shown to be emotionally involving and extremely easy to use (M. Bricken, 1991a; M. Bricken, 1991b).

There is abundant evidence that perceptual processes can assist students who are having difficulty learning abstract material. Larkin and Simon (1987) give an explanation. They argue that expressing ideas spatially, for example in diagrams, allows the information to be analyzed much more efficiently by perceptual processes than by linear cognitive processes. When a spatial representation is used to solve a problem, the spatial relations among the objects in the diagram convey important information which can be accessed and used simply by looking. Larkin and Simon demonstrated that, were the same information to be expressed propositionally in text, more cognitive resources would be needed first to find the information, and then to store it in memory while the next relevant piece of information was sought. Winn, Li and Schill (1991) found empirical support for the Larkin and Simon theory. Subjects were far quicker at answering questions about kinship when they looked at family trees than when they read corresponding text. VR uses Spatial Algebra, described below, to exploit perceptual processes in similar ways by reducing students' search and memory loads when working algebra problems. It can be expected that student understanding will improve as a result.

In addition to perceptual processes promoting the more efficient extraction of relevant information, there is also evidence that spatial representation improves the learning and retention of material. Paivio (1971, 1983) has proposed that spatial information is encoded in two different though connected memory systems. One is image-based, and information encoded in it is retrieved as mental images. The other is propositional, and information is retrieved from it verbally. Thus, material that is presented to students in a way that uses spatial arrangement to convey information is encoded twice, as images and as propositions. Information presented verbally is encoded only once, as propositions. If information cannot be retrieved using one memory system, then it might be recalled using the other. The redundancy improves recall. The work of Kulhavy and his colleagues on maps (Kulhavy, Lee, & Caterino, 1985; Schwartz, 1988; Schwartz & Kulhavy, 1987, 1988), embodied in "conjoint retention" theory, proposes similar ideas. The Spatial Algebra is likely to bring this advantage as well to learning algebra.

Finally, representing mathematical concepts concretely improves performance because it provides the student with something to interact with directly. Showing numbers as sets of objects that students can actually count makes it easier for students to solve Arithmetic word problems (Lindvall, Tamburino & Robinson, 1982). Arranging numbers to multiply into two-dimensional matrices allows students to compute the answer by counting the cells in the matrix (Carrier, Post & Heck, 1985). Indeed, a great variety of ways of making mathematical concepts more concrete have been shown to be effective in learning algebra (Shumway, 1989).

The second area of research relevant to learning in VR concerns students' construction of knowledge. Recently, educational technologists have begun to move away from developing systems, such as CAI and intelligent tutors, that teach particular content, to building "shells" that facilitate certain pedagogical strategies without specifying content. (Zucchermaglia [1991] appropriately dubs these "empty" as opposed to "filled" technologies.) They are based on the premise that students construct their own meaning by interacting with material rather than being taught something explicitly (Bransford, Sherwood, Hasselbring, Kinzer & Williams, 1990; Cognition and Technology Group, 1991; Scardamalia, 1991; Spiro & Jehng, 1990; Spiro, Feltovich, Jacobson & Coulson, 1991), and that therefore to specify a particular content organization or instructional strategy is counterproductive. These techniques are exemplified by the hypermedia system developed by Spiro and his colleagues (Spiro, Feltovich, Jacobson & Coulson, 1991; Spiro & Jehng, 1990) that lets students learn problem-solving through the exploration of ill-structured domains such as literary criticism, military strategy and, cardio-vascular medicine; and by the interactive videodisk materials developed by Bransford and the Cognition and Technology Group at Vanderbilt University (1990, 1991) that facilitate the solving of complex Mathematics problems by allowing children to interact with dramatically presented adventures.

Some key theoretical elements behind constructivism are contained in Spiro's Cognitive Complexity Theory (Spiro, Coulson, Feltovitch & Anderson, 1988; Spiro & Jehng, 1990).Cognitive Complexity Theory proposes a number of strategies to promote the acquisition of flexible knowledge. For the purposes of this project, the most important is that students "revisit the same material, at different times, in re-arranged contexts, for different purposes, and from different conceptual perspectives" (Spiro, Feltovitch, Jacobson & Coulson, 1991, p. 28). The theory sets out guidelines for achieving the appropriate amount of variability in examples in order to avoid oversimplification yet retain contact with what students already know. Students can also be guided or coached as they explore the domain. But they are always in control, determining which material to revisit on the basis of their own needs at the time.

These strategies are currently implemented in hypermedia systems which provide students with easy access to material in any order at any time. VR is even better suited to implement these strategies than hypermedia systems. When we place the student inside the world that embodies the knowledge domain, all material can be accessible, and the student can "visit" it simply by turning towards it, pointing at it, or picking it up.

The differences between VR and the hypermedia technologies used by the constructivists are that the virtual world can be programmed to obey the laws of algebra, and that the student inhabits the world and can act on it naturally. There is therefore no interface to come between the student and the world. This section describes our innovative approach to teaching algebra through virtual worlds.

The key idea of Spatial Algebra is that algebra is easier to learn and to use when it is expressed in three dimensional space rather than in one dimensional text. In Spatial Algebra, students can bring perceptual and psychomotor as well as cognitive processes to problem solving, operating directly on the objects in the virtual world. Algebraic concepts can be expressed more clearly when freed of textual manipulation and rearrangement operations. Commutativity, for example, is better represented by objects placed in arbitrary positions in space than by $X + Y = Y + X$.

Numbers and variables can be represented by three-dimensional blocks. Number blocks can have their magnitude written on them; variable blocks can be unlabelled, or labeled with variable letters, or identified by colors. We propose to determine empirically successful representational strategies.

In the Spatial Algebra, algebraic operations are mapped onto spatial configurations. To do addition, the student places blocks into the same space. Multiplication requires that blocks touch each other. Subtraction is the same as addition, except that negative quantities have a -1 multiplying block in their piles. The student divides by inverting a block. To collect terms in an equation, the student simply moves blocks from one space (or "side" of the equation) to another while adding a -1 multiplier to each stack that is moved. An alternative representation would involve normalizing the equation by having the student place all terms on one side (in the form $A = 0$). Because zero is equivalent to the void in Spatial Algebra, the students solving an equation would attempt to make all terms disappear.

Since virtual objects are programmable, pedagogical strategies can be embedded into the behavior of algebra blocks. The world serves as an experiential tutor. When the behavior of objects in the Algebra World obey the rules of algebra, actions can be classified as either "lawful" or "unlawful". For example, moving a block from one side of an equation to the other without adding the -1 multiplier leads to an unlawful situation. Substituting a 5 block for a 2 block and a 3 block is lawful. Pedagogical strategies in the Algebra World reflect the ways that the world reacts when it finds itself in an unlawful condition.

We propose to let the Algebra World act in one of three ways when an unlawful condition arises. The world can ignore the condition, and let the student figure out why the solution to a problem fails to develop. The world can correct the condition, modeling for the student what should have been done. The world can refuse to become unlawful and revert to its last lawful condition. Thus, if the student failed to add a -1 multiplier to a block that was moved from one side of the equation to the other, the world could: do nothing, forcing the student to diagnose the problem and backtrack; add the multiplier for the student, demonstrating the correct operation to make the world's condition lawful; or move the block back to where it came from, showing the student that an error had been made without precisely identifying the error. As well, the world could automatically add the multiplier serving as an algebraic calculator which relieves the student of computational details.

The programmability of Algebra World permits a great deal of pedagogical flexibility. We could automate arithmetic simplification, focusing on algebraic skills. We could permit only lawful transformations for all operations except the distributive rule, focusing on a particular algebraic skill. Pinpointing the sources of students' difficulties in this way also provides an extremely useful diagnostic tool.

Work Plan

The project has five objectives which we stated above:

1) To provide "proof of concept" of VR for teaching math;

2) To show that VR is particularly useful with less able students;

3) To develop a virtual algebra world;

4) To collaborate with teachers; and

5) To develop an experiential mathematics curriculum.

1. "Proof of concept" of VR

Attainment of this objective will require three kinds of evidence. First, we must show that students can learn algebra in VR. Several sources of data are proposed to provide this evidence. Videotapes will be made of student participants. These tapes will provide precise records of what the student experiences and of what the student does. The computer will make a trace of all operations the students perform in the virtual world. This will provide a complete record of all computations and manipulations, allowing us to assess student progress. Posttests will also be administered in VR. Student performance on test problems will indicate whether or not they have mastered algebra concepts. For each problem, we will record correctness of solution, time to solution, number of steps to solution and method of solution. Students will be expected to reach established criteria on each factor.

Second, we must show that students understand the concepts and rules of algebra (for example, commutative and distributive rules, and the steps to take to solve an equation for a variable), and not just learn algorithmic procedures. We propose two additional methods to assess the degree of understanding of algebra. First, students will be trained to "think aloud" while they are working in the algebra world. Think aloud protocols are a standard technique for assessing the development of understanding in a domain

(Ericsson & Simon, 1984). Second, students will be given algebra problems to solve that require the transfer of what they have learned in VR to other settings. Paper-and-pencil tests over the concepts they have learned will assess transfer to standard classroom tasks.

Third, we must show that students perform better after learning in VR than they do after learning from traditional methods. We will compare the performance of VR students with the performance of students who are taught with traditional classroom methods on all of the posttests described above. We plan to start working with elementary algebra students at the beginning of the 1992 school year.

2. VR and less able students

Our supposition is that VR will be especially beneficial for students who have difficulty learning from traditional methods. We shall therefore divide VR students and controls into high and low aptitude for learning algebra. Assignment to groups will be based on MAT math scores, pre-algebra math grades, and teachers' personal assessments.

3. Designing a Virtual Algebra World

Our intention is to develop and test the algebra world in the early part of and Summer of 1992. Standard instructional design procedures will be used for this (Gagne, Briggs, & Wager, 1988; Reigeluth, 1983, 1987). This will require writing instructional goals and objectives; determining student performance criteria; describing system performance, including which algebra rules to employ, and how they are to be used as pedagogical strategies; developing prototypes of the world; and testing and revising the world.

4. Collaborating with teachers

The long-term goal of the project is to improve the mastery of algebra in schools. All too often, technology-based projects have failed because they were developed in laboratories isolated from the classrooms in which they were intended to be implemented. We propose to avoid this problem by involving teachers and students in all phases of the project. Teachers will serve as subject-matter consultants, will help build the algebra world, will advise on pedagogical strategies for dealing with problems students typically experience, and will work with us and their students during the evaluation phase. We intend to consult with teachers about the most effective ways to implement VR in schools, and to interview them more formally about their experiences working with VR. These data will allow us to make recommendations concerning a wider-scale implementation of VR in schools. A number of schools and individual teachers have expressed an interest in taking part in the project.

5. Developing an experiential mathematics curriculum

We have developed spatial techniques for logic, arithmetic, and geometry, as well as for algebra. We intend to broaden the scope of this research to the entire high school mathematics curriculum, should this project justify such expansion.

Data and analysis

We shall analyze numerical data from posttests using 2 by 2 analysis of variance. Two treatments (VR and control students) will be crossed with high and low aptitude for mathematics. Interactions between aptitude and treatment for the posttest and transfer measures will indicate whether or not VR is more effective with low-aptitude students (Cronbach & Snow, 1977).

Transcripts will be made of the videotapes and think- aloud protocols. We shall analyze these using Ericsson and Simon's (1984) techniques of protocol analysis.

Researchers and collaborating teachers will keep logs of their activities. These will serve both as "audit trails" for the non-experimental data and as a means to determine the best ways of conducting similar projects in the future.

We propose the following schedule of activities for our project. We will begin Phases I and II in the early part of 1992. This will allow us to start classroom work with students beginning algebra in the Fall of 1992. We intend to iteratively refine the Algebra World and its pedagogical strategies during the 1992 school year. Data analysis and reports will be completed during the latter half of 1993.

PHASE I: INSTRUCTIONAL DESIGN (6 months)

1. Identify collaborating teachers and schools.

2. With teachers, identify a specific area within introductory algebra to be addressed by VR techniques. This content must include basic concepts, such as "variable," "constant," "equation," and the basic algebraic operations.

3. Write goals and objectives. Specify scope and sequence of curricular material to cover.

4. Identify instructional methods for comparison (non-VR) groups. Develop testing materials.

PHASE II: VIRTUAL WORLD DESIGN (9 months, overlapping Phase I)

1. Develop the virtual environment operating system and associated software and hardware.

2. With teachers, design the virtual algebra world. Identify unique instructional aspects of virtual world.

3. Implement the virtual algebra world, with sample of first-year algebra students for pilot testing, iterating development to meet instructional objectives.

PHASE III: EXPERIMENTATION (9 months)

1. Select sample of beginning algebra students.

2. From MAT scores, course grades, and teachers identify those who are strong and weak in math. Teach any lacking prerequisite skills and knowledge for entry into the algebra world.

3. Place students in virtual algebra world.
   a. Familiarize student with VR experience and with thinking aloud.
   b. Allow free exploration of the world.
   c. Assign a task, depending on the current objective (Examples of task: explore and manipulate the objects in the world, factor an expression, solve an equation).
   d. Gather video, computational traces of activity, and think-aloud data.
   e. Debrief student.

4. Administer post and transfer tests.

5. Analyze data.
   a. Analyze think aloud protocols.
   b. Determine whether students have mastered content.
   c. Interpret student success in terms of student aptitude.

6. Carry out iterative revision from task 3.
   a. Modify instructional strategies.
   b. Modify virtual world.

7. Test for transfer to academic and "real-world" knowledge and skills.

8. Compare results from VR experiences to results from comparison groups.
   a. Analyze and abstract all research results.
   b. Publicly distribute refined Algebra World.
   c. Write reports.

We expect to share the results of our project through the normal publication channels. Interest in VR among academic and professional colleagues has already secured a presentation at the annual meeting of the American Educational Research Association in San Francisco in April 1992. Both PI's are active in professional associations, including AERA, AECT, ADCIS, ACM, SIGGRAPH, and publish regularly in scholarly and professional journals. In addition, it is our intention to place the software we develop into the public domain. There is a great deal of public, professional, and academic interest in VR and we expect our project to attract a great deal of attention.

Attachment A: Spatial Algebra

References

Austin, J.D., & Vollrath, H.J. (1989). Representing, solving and using algebraic expressions. Mathematics Teacher, 82 (8), 608-612.

Berman, B., & Friederwitzer, F. (1989). Algebra can be elementary ... When it's concrete. Arithmetic Teacher, 36 (8), 21-24.

Bernard, J.E., & Bright, G.W. (1982). Student performance solving linear equations. Proceedings of the Sixth International Conference for the Psychology of Mathematical Education, 144-149.

Bransford, J.D., Sherwood, R.D., Hasselbring, T.S., Kinzer, C.K., & Williams, S. (1990). Anchored instruction: Why we need it and how technology can help. In D. Nix & R. Spiro (Eds.), Cognition, education and multimedia: Exploring ideas in high technology. Hillsdale, NJ: Lawrence Erlbaum Associates.

Bricken, M. (1991a) Virtual Worlds: No Interface to Design. in M. Benedikt, Cyberspace: The First Steps, MIT Press, Cambridge, MA.

Bricken, M. (1991b). Virtual reality learning environments: potentials and challenges. Computer Graphics Magazine, ACM, 7/91

Bricken, W.M. (1987). Analyzing errors in elementary mathematics. Doctoral dissertation, Stanford University.

Bricken, W.M. (1990). Virtual environment operating system: preliminary functional architecture. HITL Technical Report M-90-2, University of Washington.

Bright, G.W. (1981, April). Student procedures in solving equations. Paper presented at the Annual Meeting of the National Council of Teachers of Mathematics, St. Louis, MO.

Carrier, C., Post, T.R., & Heck, W. (1985). Using microcomputers with fourth-grade students to reinforce arithmetic skills. Journal for Research in Mathematics Education, 16, 45-51.

Cognition and Technology Group at Vanderbilt University (1991). Technology and the design of generative learning environments. Educational Technology, 31, (5), 34-40.

Connell, M.L, & Ravlin, S.B. (1988, April). The flagpole factory: Providing a referent for linear equations. Paper presented at the Meeting of the International Association for Computers in Education, New Orleans, LA.

Cronbach, L.J., & Snow, R. (1977). Aptitudes and instructional methods. New York: Irvington.

Ericsson, K.A., & Simon, H.A. (1984). Protocol Analysis. Cambridge, MA: MIT Press.

Gagne, R.M., Briggs, L.J., & Wager, W.W. (1988). Principles of instructional design: Third edition. New York: Holt Rinehart and Winston.

Gerace, W.J., & Mestre, J.P. (1982). The learning of algebra by 9th. graders: Research findings relevant to teacher training & classroom practice. Final Report, National Institutes of Health, Washington DC, (Contract # 400-81-0027).

Greeno, J. (1985). Investigations of a cognitive skill. Technical Report. Pittsburgh: University of Pittsburgh Learning and Development Center.

Kaput, J.J. (1978). Mathematics and learning: Roots of epistemological status. In J. Lochhead & J. Clements (Eds.), Cognitive process instruction. Philadelphia, PA: Franklin Institute Press.

Kulhavy, R.W., Lee, J.B., & Caterino, L.C. (1985). Conjoint retention of maps and related discourse. Contemporary Educational Psychology, 10, 28-37.

Larkin, J.H., & Simon, H.A. (1987). Why a diagram is (sometimes) worth ten thousand words. Cognitive Science, 11, 65-99.

Lindvall, C.M., Tamburino, J.L., & Robinson, L. (1982, March). An exploratory investigation of the effect of teaching primary grade children to use specific problem-solving strategies in solving simple story problems. Annual Meeting of the American Educational Research Association, New York.

O'Shea, T. (1986, April). Students' understanding of basic algebra concepts. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

Paivio, A. (1971). Imagery and verbal processes. New York: Holt, Rinehart and Winston.

Paivio, A. (1983). The empirical case for dual coding. In J.C. Yuille (Ed.). Imagery, memory and cognition. Hillsdale: Lawrence Erlbaum, 310-332.

Reigeluth, C.M. (1983). Instructional design: What is it and why is it? In C.M. Reigeluth (Ed.), Instructional design theories and models. Hillsdale, NJ: Lawrence Erlbaum Associates.

Reigeluth, C.M. (1987). Instructional theories in action. Hillsdale, NJ: Lawrence Erlbaum Associates.

Rosnick, P. (1980). The presentation of the concept of variable and the development of problem solving skills: A multi-text review. Report, Washington DC: National Science Foundation, (Contract # SED80-16567).

Rosnick, P. (1981). Some misconceptions concerning the concept of variable. Mathematics Teacher, 74, 418-420.

Scardamalia, M. (1991, April). Computer environment for group-based knowledge construction: CSILE (Computer-Supported Intentional Learning Environments). Paper presented at the the Annual Meeting of the American Educational Research Association, Chicago.

Schwartz, N.H. (1988). Cognitive processing characteristics of maps: Implications for instruction. Educational and Psychological Research, 8, 93-101.

Schwartz, N.H., & Kulhavy, R.W. (1987). Map structure and the comprehension of prose. Educational and Psychological Research, 7, 113-128.

Schwartz, N.H., & Kulhavy, R.W. (1988). Encoding tactics in the retention of maps. Contemporary Educational Psychology, 13, 72-85.

Shumway, R.J. (1989). Solving equations today. School Science and Mathematics, 89, 208-219.

Sleeman, D.H. (1984). Solving linear algebraic equations. Mathematics in school, 13, (4), 37-38.

Sowell, E.J. (1989). Effects of manipulative material in Mathematics instruction. Journal of Research in Mathematics Education, 20, 498-505.

Spiro, R.J., Coulson, R.L., Feltovich, P.J., & Anderson, D.K. (1988). Cognitive flexibility theory: Advanced knowledge acquisition in ill-structured domains. In Proceedings of the Tenth Annual Conference of the Cognitive Science Society, Hillsdale, NJ: Lawrence Erlbaum Associates.

Spiro, R.J., Feltovich, P.J., Coulson, R.L, & Anderson, D.K. (1989). Multiple analogies for complex concepts: Antidotes to analogy-induced misconception in advanced knowledge acquition. In S. Vosniadou & A. Ortony (Eds.) Similarity and analogical reasoning. Cambridge: Cambridge University Press.

Spiro, R.J., Feltovich, P.L., Jacobson, M.J., & Coulson, R.L. (1991). Cognitive flexibility, constructivism and hypertext: Random access instruction for advanced knowledge acquisition in ill-structured domains. Educational Technology, 31 (5), 24-33.

Spiro, R.J., & Jehng, J-C. (1990). Cognitive flexibility and hypertext: Theory and technology for the nonlinear and multidimensional traversal of complex subject matter. In D. Nix & R. Spiro (Eds.), Cognition, education and multimedia: Exploring ideas in high technology. Hillsdale, NJ: Lawrence Erlbaum Associates.

Thwaites, G.N. (1982). Why do children find algebra difficult? Mathematics in school, 11, (4), 16-19.

Wagner, S. (1981). Conservation of equation and function under transformation of variable. Journal for Research in Mathematics Education, 12, 107-118.

Waits, B.K., & Demana, F. (1989). A computer-graphing-based approach to solving inequalities. Mathematics Teacher, 85, 327-331.

Watkins, W., & Taylor, M. (1989). A spreadsheet in the Mathematics classroom. Collegiate Microcomputer, 7, 233-239.

Winn, W.D., Li, T-Z., & Schill, D.E. (1991). Diagrams as aids to problem solving: Their role in facilitating search and computation. Educational Technology Research and Development, 39, 17-29.

Zehavi, N., & Bruckheimer, M. (1984). (x - y) squared is equal to x squared - y squared. Mathematics in School, 13, (1), 22-23.

Zucchermaglia, C. (1991, May). Toward a cognitive ergonomics of educational technology. Paper presented at the NATO Advanced Research Workshop on The Design of Constructivist Learning Environments. Leuven, Belgium.