SPATIAL REPRESENTATION of ELEMENTARY ALGEBRA

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Abstract

Our understanding of a concept is tightly connected to the way we represent that concept. Traditionally, mathematics is presented textually. As a consequence novice errors, in elementary algebra for example, are due as much to misunderstandings of the nature of tokens as they are to miscomprehensions of the mathematical ideas represented by the tokens. This paper outlines a *spatial algebra* by mapping the structure of commutative groups onto the structure of space. We interact with spatial representations through natural behavior in an inclusive environment. When the environment enforces the transformational invariants of algebra, the spatial representation affords experiential learning. *Experiential algebra* permits algebraic proof through direct manipulation and can be readily implemented in virtual reality. The techniques used to create spatial algebra lay a foundation for the exploration of experiential learning of mathematics in virtual environments.

1. Introduction

How we think about mathematical concepts is often constrained by our representation of those concepts. Syntax and semantics (representation and concept) are tightly connected. The addition operation, for example, is conceptualized as binary when written in linear text:

$$x + y$$

To add three numbers, we must use two addition operators:

$$x + y + z$$

Column addition, however, reconceptualizes the addition operation to be variary (one operator can be applied to an *arbitrary* number of arguments):

```
   w
   x
   y
+  z
----
```

Naturally, the addition algorithms and techniques taught to students differ for the different representations.

The traditional representation of binary addition is one-dimensional. There are two locations for arguments, one on either side of the textual operator. Column addition increases the dimension of representation to the plane; digits of individual numbers are expressed horizontally, different numbers are expressed vertically. From a spatial perspective, the number of arguments that can be added in one operation depends upon the dimension of the representation.

In general, how we represent numbers is a matter of convenience. For learning mathematics (and for doing mathematics) it is often more convenient to call upon visual interaction and natural behavior than it is to conduct symbolic substitutions devoid of meaning. *Spatial algebra* uses the three dimensions of natural space to express algebraic concepts. A higher dimension of representation greatly simplifies the visualization and the application of algebraic axioms. Algebraic transformation and the process of proof are achieved through direct manipulation of the three-dimensional representation of the algebra problem.

The difficulties children have when they begin to learn algebra are well documented [9] [7] [17] [8] [4]. Spatial algebra addresses common errors made by novice algebra students by permitting experiential interaction with abstract representations. Spatial representations enhance understanding [11]. Concrete manipulation is known to be an effective teaching technique [15] [1] [14].

Virtual reality is a computer generated, multi-dimensional, inclusive environment which can be accepted by a participant as cognitively valid [6]. VR teaching systems overcome the inconvenience of an insufficiently abstract physical reality by combining mathematical abstraction with the intuition of natural behavior. The programmability of VR allows a curriculum designer to embed pedagogical strategies into the behavior of virtual objects which represent mathematical structures [2]. Using a VR presentation system, the axioms of algebra can be, so to speak, built into the behavior of the world.

The visual programming community has developed taxonomies of visual approaches [13]. The experiential approach to mathematical formalism presented in this paper is sufficiently unique not to fit into existing taxonomies of visual languages. The approach of mapping formal operations onto the topological structure of space itself is not diagrammatic, iconic, or form-based. Most fundamentally, experiential mathematics imparts semantics onto the void (empty space). Actively using the void is both simple and conceptually treacherous [3]. The spatial techniques in this paper are general and have been applied to several formal systems, including elementary logic and integer arithmetic [3] [5].

2. Spatial Algebra

The components of space which can be used for the representation of mathematical concepts include:

- -- empty space (the void),
- -- partitions between spaces (boundaries, objects),
- -- labeled objects which share a space, and
- -- labeled objects which share a boundary (touch one another).

This is sufficient structure for the expression of elementary algebra. One possible map from algebraic tokens to algebraic spaces is:

Constants:			
	{ 1,2,3,...}	-->	{ labeled-blocks }
Variables:			
	{ x,y,z,...}	-->	{ labeled-blocks }
Operators:			
	{ + }	-->	{ sharing-space }
	{ * }	-->	{ sharing boundaries }
Relations:			
	{ = }	-->	{ partitions of space }

Examples of a spatial representation of the above map follow. The appendix to this paper contains a list of principles for designing spatial representations.

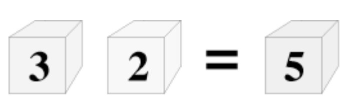
Constant as labeled block:

3

Variable as labeled block:

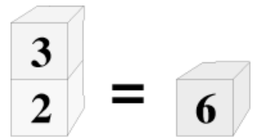
x

Space sharing as *addition*:



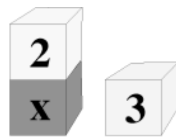
3 + 2 = 5

Touching as *multiplication*:



3 * 2 = 6

A simple algebraic term:



2x + 3

The gravitational orientation of the typography (top to bottom of page) in the above examples is not an aspect of spatial algebra, although gravitational metaphors are useful for the representation of sequential concepts such as non-commutativity. As well, the sequencing implied by stacked blocks is an artifact of typography; stacks only represent groups of objects touching in space.

3. Group Structure of Spatial Forms

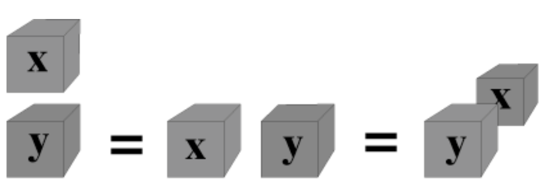
Generally, spatial representation can be mapped onto group theory. A commutative group is a mathematical structure consisting of a set and an operator on elements of that set, with the following properties:

-- The set is closed under the operation.
-- The operation is associative and commutative.
-- There is an identity element.
-- Every element has an inverse.

The integer addition and multiplication operators taught in elementary school belong to the commutative group.

3.1 Commutativity

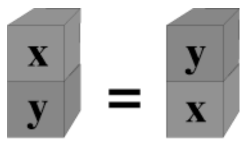
Spatial representation permits the implicit embedding of commutativity in space. The *commutativity of addition* is represented by the absence of linear ordering of blocks in space (visualize the blocks in this example as floating in space rather than in a particular linear order):



x + y = y + x

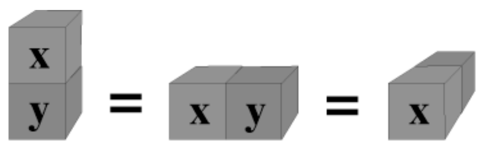
We intuitively recognize objects contained in a three-dimensional space as ordered solely by our personal perspectives. In contrast, typographical objects are necessarily ordered in sequence by the one-dimensional nature of text and by the two-dimensional nature of the page.

Commutativity of multiplication can be seen as the absence of ordering in touching blocks:



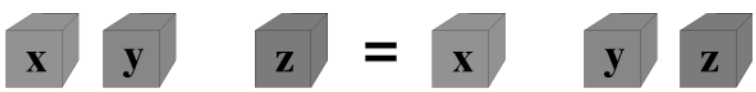
$$x * y = y * x$$

Again, in space there is no preferential ordering to touching objects:



3.2 Associativity

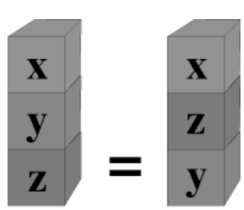
Associativity of addition is the absence of an explicit grouping concept in space:



$$(x + y) + z = x + (y + z)$$

The apparent visual grouping expressed by differences in metric distance between blocks can be assigned a semantics of associativity (for example, add closest objects first), or it can be ignored, permitting the operation assigned to space to address multiple arguments in parallel. From an intuitive perspective, operations embedded in space apply to any number of objects in that space. Whatever grouping we use is a matter a choice and convenience. Parallel computers provide techniques for addressing all objects at the same time.

Associativity of multiplication is the absence of an explicit grouping concept in piles:

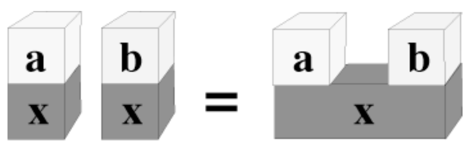


$$(x * y) * z = (x * z) * y$$

The apparent visual ordering of piles can be overcome by assuming that all objects in a pile touch one another directly. Rather than displaying stacked objects, VR might present objects in piles as completely interpenetrating. Every object in this non-physical representation is in contact with every other object, forming a Cartesian product of touching objects.

3.3 Distribution

Precedence operations associated with the distributive rule are the most common algebraic error for first year students [12] [4]. The representation of distribution in spatial algebra is particularly compelling. Generally, the distributive law permits combining blocks with identical labels into a single block with that label. Conversely (read right to left), *distribution* permits splitting a single block that touches separate piles into separate but identical blocks touching each pile:



$ax + bx = (a + b)x$

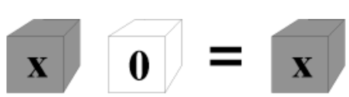
Blocks with identical labels are both singular and arbitrarily subdividable in space. This ability to arbitrarily divide and combine blocks with a common name is the same as the ability to arbitrarily create duplicate labels in a textual representation. Changing the size and the number of occurrences of a labeled block is easy in a virtual environment.

Any potential ambiguity between distributive idempotency and the use of space as the addition operator is avoided by the effect of *context* on interpretation. Idempotency requires the context of touching blocks (multiplication). Addition requires the context of non-touching piles.

3.4 Identities

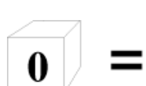
Zero is the identity element for addition. The identity in the spatial metaphor is the void; identities are equivalent to empty space.

The *additive identity*:

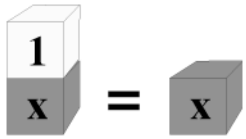


$x + 0 = x$

That is, zero disappears in space:

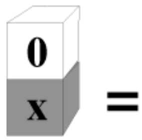


The *multiplicative identity*:



1 * x = x

The One block disappears only in the context of an existing pile. A zero in a pile makes the entire pile disappear:

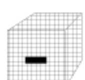


0 * x = 0

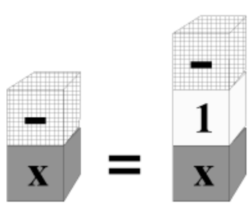
3.5 Additive Inverse

The inverse of a positive number is a negative number. *Negative numbers are the most difficult aspect of arithmetic for elementary students.* One way to directly represent inversion is to create an *inverter block*. Another way is to create an inversion space; for example using "under-the-table" for inverses. Inverses can be represented in many ways: as inverters, as colors, as orientations, as different spaces, as binary switches, as dividing planes, as inside-out objects.

In this version of spatial algebra, piles are inverted by the inclusion of a special inverter block:



Since a negative number can be seen as being multiplied by -1, the inverter block is expressed as touching (multiplying) the pile which is inverted:

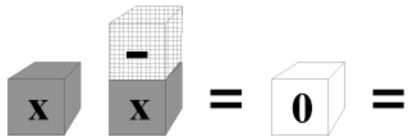


-x = (-1) * x

The inverter block expresses subtraction as the addition of inverses,

x - x   is written as   x + (-x)

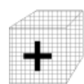
The *additive inverse*:



$x + (-x) = 0$

3.6 Calculus of Signs

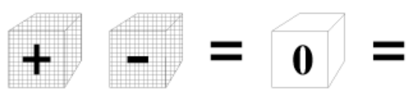
The use of the inverter block for negative numbers introduces a calculus of signs into the algebra of integers. A sign calculus requires the explicit introduction of the positive block:



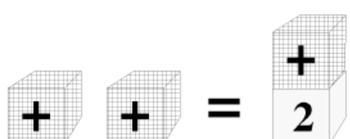
The positive block is the inverse of the inverter block. It introduces the concept of polarity and the act of cancellation. Numbers without signs are usually assumed to be positive. Making signs explicit removes this assumption.

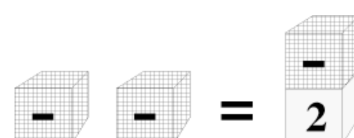
The following rules of sign calculus assume each sign has a unit value associated with it.

Additive cancellation in space:

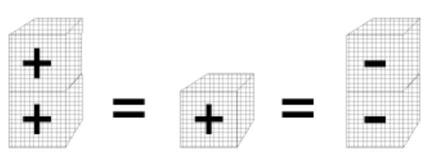


Cardinality in space:

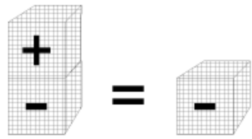




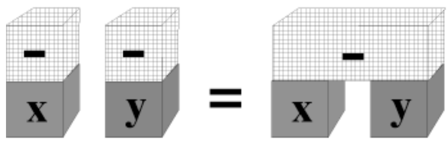
Multiplicative cancellation in piles:



Multiplicative dominance in piles:



The following example illustrates an inverter sign distributed across all objects in a space:



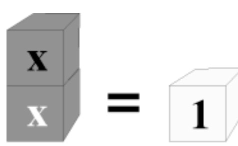
(-x) - y = -(x + y)

3.7 Multiplicative Inverse

Finally, division is the multiplicative inverse. Again, there are many possible ways to represent an inverse in a spatial representation. Since the traditional notation for fractions is primarily two-dimensional, it already has many spatial aspects. The division line that separates numerator from denominator could be carried over to the spatial representation as a plane dividing a pile into two parts. Here however, the multiplicative inverse is represented by inverse shading of the block label:

1/x

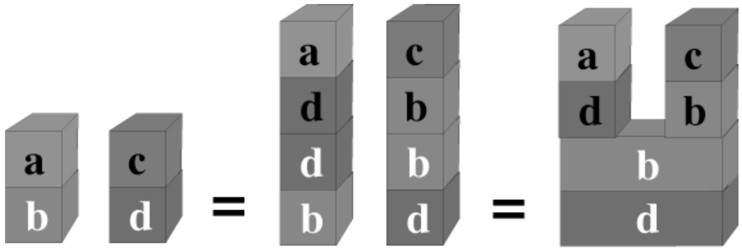
The *multiplicative inverse*:



x * 1/x = 1

One weakness with the choice to represent a reciprocal as differently shaded labels is that composition of reciprocals -- for example 1/(1/x) -- is not visually defined. Choice of representation necessarily effects pedagogy. It is an empirical question as to which representations facilitate learning algebraic concepts efficiently.

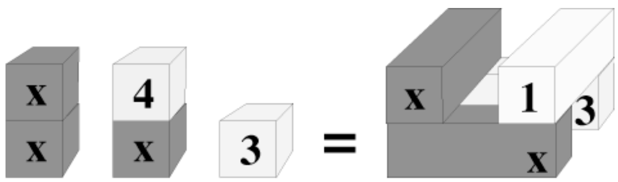
Fractions are the second most difficult area for students of arithmetic. A typical problem using fractions requires the application of the distributive rule:



$a/b + c/d = (ad + bc)/bd$

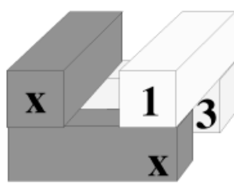
4. Factoring

Factoring polynomial expressions is equivalent to multiple applications of distribution. For instance:



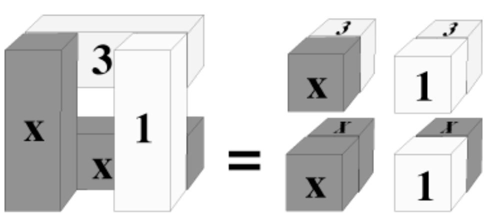
$x^2 + 4x + 3 = (x + 1)*(x + 3)$

One advantage of the spatial representation on the right-hand-side of this equation is that both the factored and the polynomial forms are visible concurrently. Looking from the side, we see two completely touching spaces which represent the factored form:



$(x + 1) * (x + 3)$

Looking down from the top, we see four piles which represent the polynomial form:



$x^2 + 1*x + 3*x + 1*3$

Here, the factored form is converted to the polynomial by slicing each addition space through the middle.

5. Caveats

Experiential mathematics is quite new as a formalism. The idea of mapping semantics onto the void first appeared in a mathematical text that is widely acknowledged as impenetrable [16]. Spatial algebra is an interpretation of the abstract mathematics developed by Spencer-Brown in *Laws of Form* and by Louis Kauffman at the University of Chicago [10].

The representational details of the spatial algebra presented here are, like any choice of syntax, somewhat arbitrary. The text lists many options, for example, for the representation of inverses. This representational freedom can be constrained by empirical studies intended to determine which particular representations are effective for task performance. There is no reason to believe that effective representations are generic. More probably, different individuals will prefer and understand different representations in the context of different tasks. One strength of VR is that it is completely customizable to individual participants. Still, the research to determine which representations are effective has yet to be conducted. In fact, demonstrating that spatial algebra actually improves performance in high school algebra remains as future research.

Significant components of a complete spatial mathematics have not been included in this paper. In particular, a compelling representation for exponentiation is missing. Spatial arithmetic has been assumed. A technical refinement is needed for the calculus of signs, to either remove cardinality completely from signs, or embed it deeper, expressing "-" as "-1". We also need to consider representation of functions such as the logarithm and sine. Spatial solutions to these shortfalls exist, but a completely integrated spatial mathematics is not yet formulated.

The weakest aspect of the proposed spatial algebra is the representation of three or more multiplied objects, x*y*z for example. This form can be represented by either completely interpenetrating blocks or by "blocks" with complex shapes that twist around to touch all other blocks. This problem gets particularly difficult for multiplying several factored expressions, for example: (x + 1)*(x + 2)*(x + 3)

In general, the cubic blocks presented in this paper are misleading, since they imply a Cartesian coordinate system. In fact, the spatial representation proposed here has no associated metric (or rather, the metric is irrelevant to the mathematical formalism). The treatment of space might be improved by explicitly including a representation for the table which blocks can be imagined to rest upon.

We also have little experience with animation of and interaction with spatial forms in VR. This paper presents the design phase of a wider study into the utility of VR for mathematics education[18].

6. Conclusion

Spatial representation provides a map to a wide range of new visual languages. The examples in this paper are expressed in a language of labeled blocks. The spatial rules, however, map just as easily onto people in a room, toys in a box, salmon in streams, and bricks in a wall.

The techniques of spatial algebra and the display capabilities of virtual environments have coevolved. Spatial algebra is proposed as an experimental approach for exploring the representation-dependent aspects of novice algebra errors. Virtual reality display systems are proposed as a straightforward way to present spatial algebra as an experiential mathematical system. During the next phase of this work, we will explore the pedagogical characteristics of spatial representations in virtual reality.

7. Appendix: Principles of Spatial Mathematics

I use the term *boundary mathematics* to describe the collection of rules and tools used to generate representations of spatial algebra. Boundary mathematics is general in that its principles can be applied to many mathematical domains. This paper, for instance, has implicitly assumed a model of spatial integers.

The roots of boundary mathematics can be found in G. Spencer-Brown's mathematical text *Laws of Form*. Boundary mathematics is quite unique, since it incorporates both the participant and the void into its formal structure. This makes formal theorems sound somewhat like pop psychology.

General Principles

1. Mathematics is the experience of abstraction.

2. Experience is not a recording. Representation is not reality.

3. The void cannot be represented.

4. Space requires participation. To participate is to partition space, to construct a boundary.

5. Boundaries both separate and connect.

6. Boundaries identify an intentional construction.

7. Representation and meaning are different sides of the same boundary.

8. Our body is our interface.

Mathematical Principles

9. Operators, invariants, and identities can be embedded in space.

10. Multiplicity is generated by observation.

11. Commutativity is embedded in space, ordering is embedded in time. All virtual entities are asynchronous parallel processes.

12. Associativity is the choice of the participant. All entities are autonomous.

13. Entities are both singular and plural in form, depending upon the construction of the participant. Entities with the same name are the same entity.

14. That which is common to every entity in a space is common to the space itself, forming the ground of the space.

15. Touching spaces are in pervasive contact (Cartesian product).

16. Crossing a boundary inverts a space. Inversion unites partitioned spaces.

17. Normalized spaces are those equivalent to the void. They can support arbitrary grounds.

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