RECLAIMING MEANING IN MATHEMATICS

A Presentation for the
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FOR TEACHERS

An educational chasm in mathematics occurs when students change learning styles from ***concrete manipulatives*** to ***abstract symbols***.

Students learn through meaningful experience.

The way ideas are conveyed makes a difference.

The concepts of mathematics can be presented using formal representations ***that are sensitive to human needs***.

Spatial mathematics connects ***number sense*** to the formal structure of mathematics.

THEME

Toward humane formal mathematics

I. HOW MEANING HAS BEEN LOST

- Separating meaning from structure
- Quality of representation
- Cognitive effort

II. FOUR TYPES OF SPATIAL MATH

Spatial algebra	(slides)
Unit-ensemble arithmetic	(math theory)
Depth-value notation	(video)
Spatial arithmetic	(demonstration)

MEANING

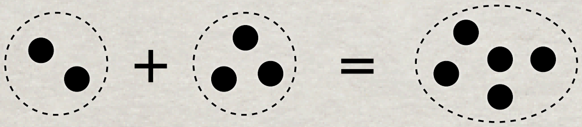
MEANING IN ARITHMETIC

What do the objects and operations of arithmetic ***mean***?

OBJECTS: integers name ensembles of identical units

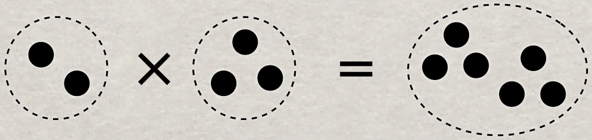
unit ensembles

ADDITION: put ensembles together in the same space



fusion

MULTIPLICATION: replace units by ensembles



substitution

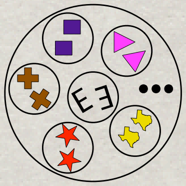
Loss of Meaning

Objects:

integers name the set of sets with the same cardinality

... −1 0 1 2 3 4 ...

2 =



Addition:

memorize rules for digits (number facts)

learn rules of position (align and carry)

$$2 + 3 = 5$$

$$\begin{array}{r} 56 \\ \underline{+\ 78} \\ \underline{\underline{134}} \end{array}$$

Multiplication:

memorize rules for digits (number facts)

learn to add while multiplying

$$2 \times 3 = 6$$

$$\begin{array}{r} 56 \\ \underline{\times\quad 78} \\ 448 \\ \underline{+\ 394\ \ } \\ \underline{\underline{4388}} \end{array}$$

HILBERT'S PROGRAM

Separate mathematics and logic from spatial intuition.

"Mathematics is a game played according to simple rules
with meaningless marks on paper." *David Hilbert (c. 1900)*

Formal structure: a finite sequence of signs, without:

- intuition
- visualization
- physical interaction
- parallelism

The rules of algebra are structural.

Group theory is about notation.

TOKENS ARE A PROBLEM

The ***current style*** of mathematical expression
is inherently difficult to understand.

$$2(x - 3(x - (2y + 1))) - 4(3(y + 1) - x) + 6$$

Mathematical ideas are represented by *strings of tokens*.
Token-strings bear no resemblance to their meaning.
Icons, in contrast, look somewhat like what they represent.

Some problems with the formal language of tokens:

- neither intuitive nor natural
- must be memorized rather than experienced
- includes misleading structural redundancy
- cannot represent concepts
- makes people think they do not understand

DISPLAY MEDIA

A VARIETY OF MEDIA

Different display media provide different types of structure, each with ***different properties***.

Clay tablets and pebbles

- unit ensembles
- physical correspondence
- concrete and constructive

Hilbert's signs

Pencil and paper (chalk and board)

- token-strings
- axiomatic correspondence
- abstract and algorithmic

19th century reality

Digital display

- icons, pictures, animations
- virtual correspondence
- both concrete and abstract

21st century reality

QUALITIES OF FORM

Some display media convey meaning more ***effectively***.

- more expressive
- less cognitive effort
- simpler algorithms
- visual, aural, tactile, experiential

"house"

actual house

Mathematical concepts, too, support a diversity of structural representations and rules.

QUALITY I: EASY

Some representations require ***less effort***.

completely new rules

FRACTIONS: $$\frac{1}{4} + \frac{1}{5} = \frac{5}{20} + \frac{4}{20} = \frac{5+4}{20} = \frac{9}{20}$$

two ***different*** notations
with different rules

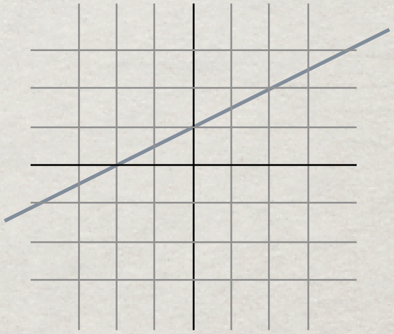
DECIMALS: $.25 + .20 = .45$

little additional effort

QUALITY II: VISUAL

Some representations are ***more visual***.

COORDINATE GRAPH:



visual

two ***very different*** notations
with different properties

LINEAR EQUATION: $y = 1/2\ x + 1$

abstract
and visual

two ***similar*** notations
with different properties

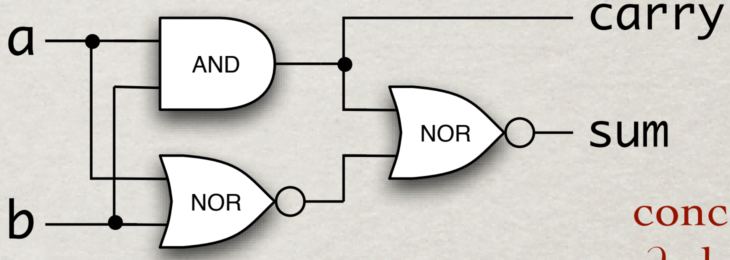
GENERAL EQUATION: $-x + 2y - 2 = 0$

abstract

QUALITY III: PHYSICAL

Some representations are ***physically manifest***.

SILICON CIRCUITRY:



concrete
and abstract

two abstract notations,
one maps to the ***physical***

BOOLEAN ALGEBRA:

$\text{sum} = a \neq b$
$\text{carry} = a \times b$

symbolic
and abstract

two abstract notations,
one maps to the ***linguistic***

PROPOSITIONAL LOGIC:

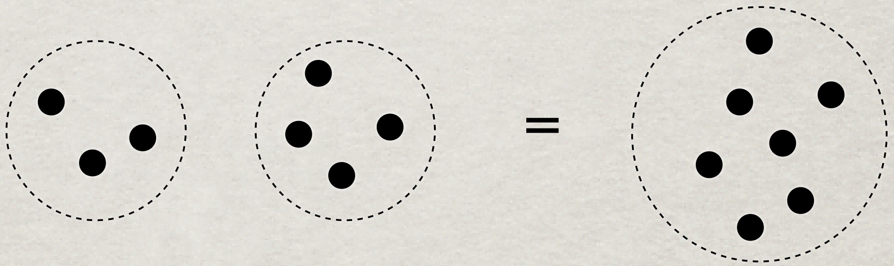
sum IFF EITHER a OR b
carry IFF a AND b

linguistic
and abstract

QUALITY IIII: SIMPLE

Some representations support ***simpler operations***.

PHYSICAL ACTION:



two different ***activities***,
one physical and one cognitive

interactivity

SYMBOLIC THOUGHT:

3 + 4

rote memory

SPATIAL MATHEMATICS

Spatial patterns are a ***formal alternative*** to token-strings.

ALGEBRA OF STRINGS:

{partitioned set-of-tokens: token-tuples $\longrightarrow$ tokens}

ALGEBRA OF SPATIAL PATTERNS:

{partitioned set-of-patterns: patterns $\longrightarrow$ patterns}

does not include the
concept of ***arity***

Spatial forms are intuitive, visual, interactive, simple.
Spatial axioms and algorithms are simple yet rigorous.

FOUR VARIETIES

Spatial Algebra with Blocks

- how to map algebraic properties onto spatial presence
- compare to *group theoretic token-strings*

Unit-ensemble Arithmetic

- how to return meaning to arithmetic
- compare to *token-based integer arithmetic*

Depth-value Notation

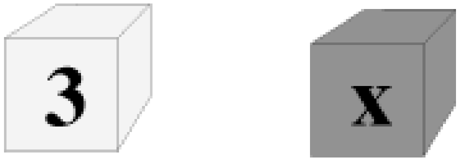
- how to make meaningful arithmetic simple
- compare to *place-value notation*

Spatial Arithmetic with Blocks

- how to provide physical, interactive calculation
- compare to *symbolic arithmetic*

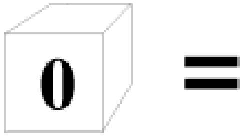
SPATIAL ALGEBRA WITH BLOCKS

SPATIAL ALGEBRA
FACTS



3 $\quad$ x

numerals and variables are
BLOCKS



0

additive zero
is VOID

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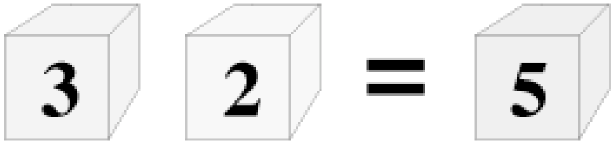
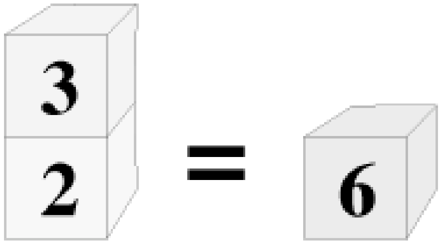
group theory ⟶ spatial presence



$3 + 2 = 5$

addition is
SHARING SPACE

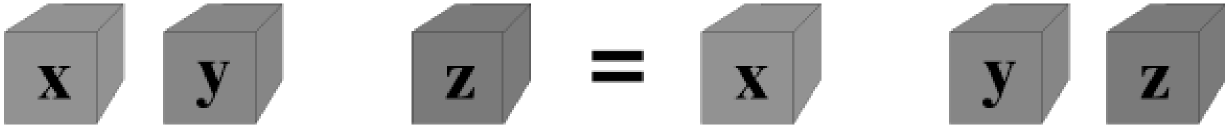


$3 \times 2 = 6$

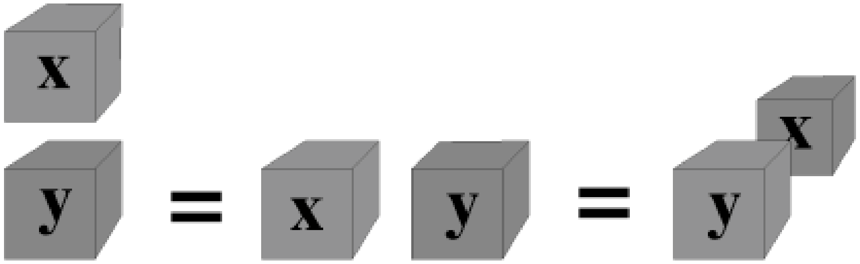
multiplication is
TOUCHING

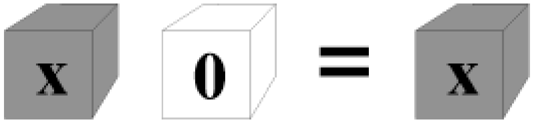
SPATIAL ALGEBRA

ADDITION



associativity and commutativity are SHARING SPACE



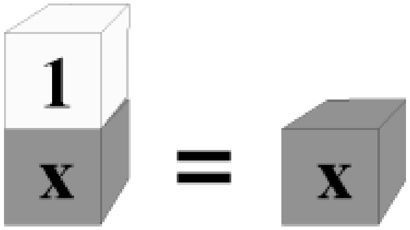


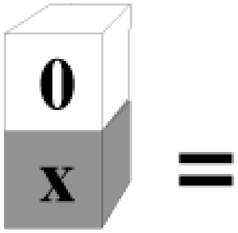
add-zero is SHARING SPACE with nothing

SPATIAL ALGEBRA

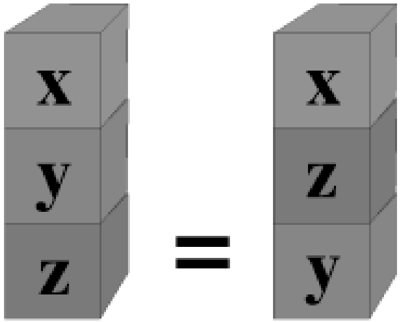
MULTIPLICATION

BLOCKS are unitary

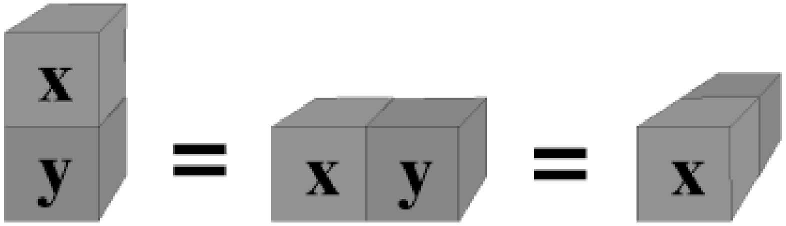




TOUCHING explicit
void annihilates

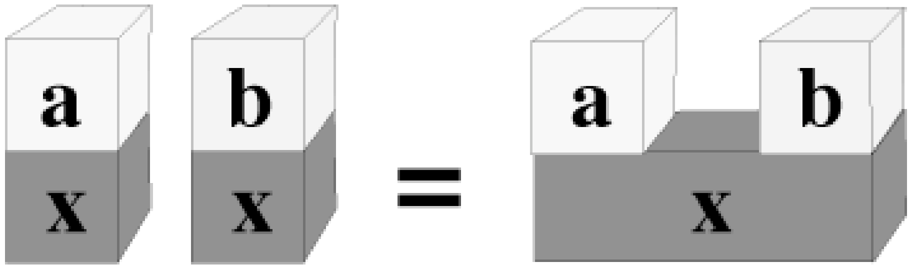


associativity and commutativity are
TOUCHING

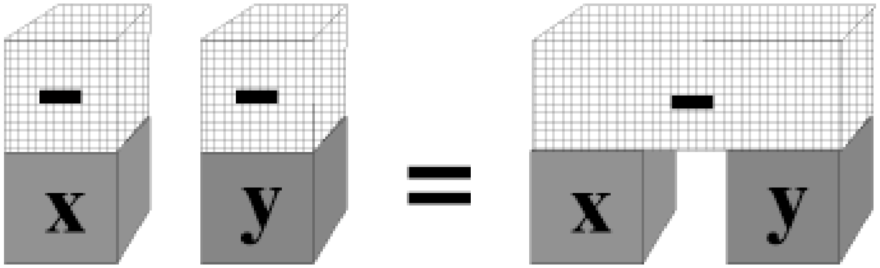


SPATIAL ALGEBRA

DISTRIBUTION

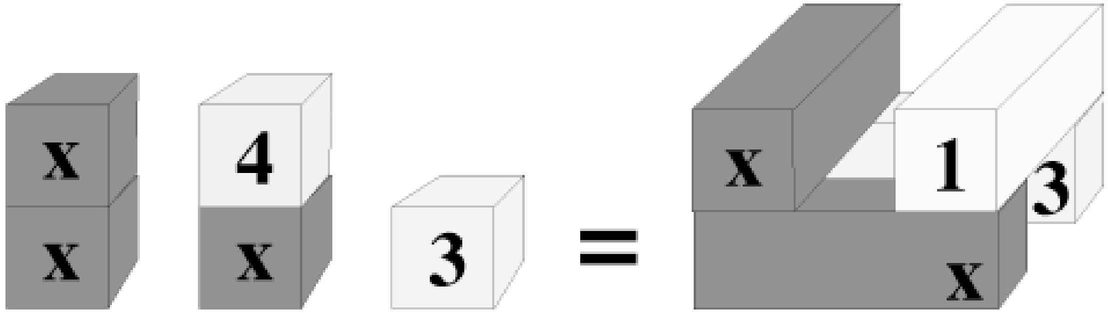


distribution is SLICING or JOINING identical blocks

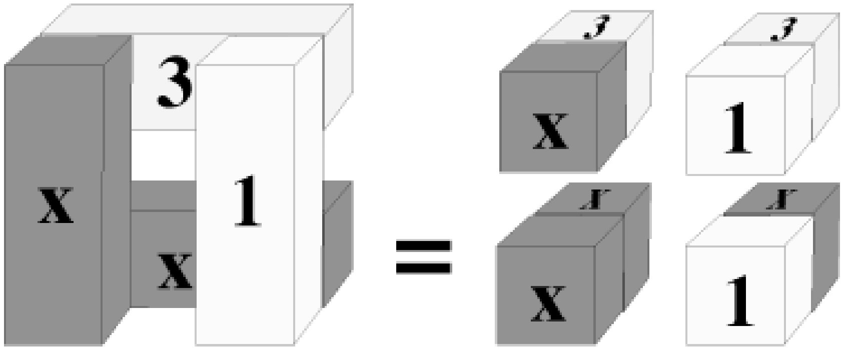


SPATIAL ALGEBRA

FACTORING

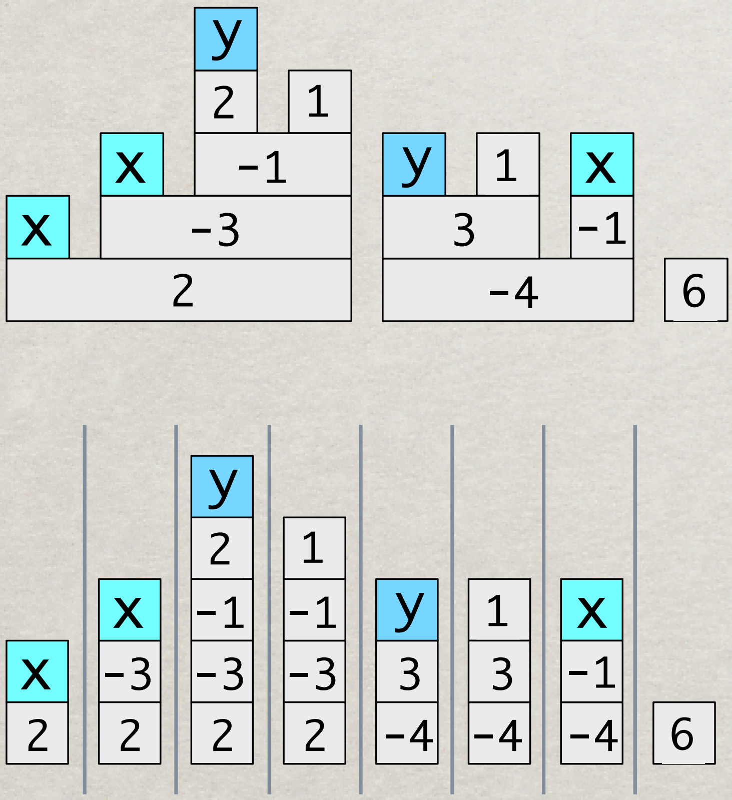


polynomial forms are SLICED factored forms

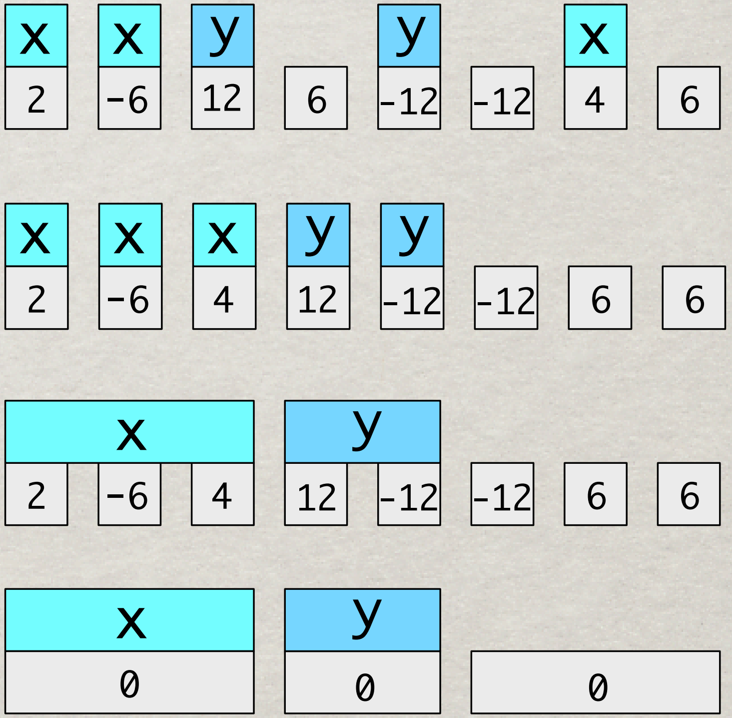


DISTRIBUTION IN DEPTH

2(x – 3(x – (2y + 1))) – 4(3(y + 1) – x) + 6 = 0



assume number facts



UNIT-ENSEMBLE ARITHMETIC

UNIT ARITHMETIC

The ***simplest arithmetic*** is based on identical units: fingers, pebbles, shells, marks, strokes, or tallies.

Tally sticks were in use 30,000 years ago.

Sumerian numerals are over 5,000 years old.

Unit-ensembles are groupings of units without specific names.

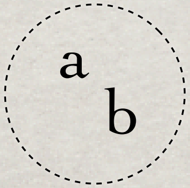
- base-1, units are indistinguishable
- one-to-one correspondence without counting
- add by putting together (additive principle)
- often considered to be the *definition* of whole numbers

UNIT ADDITION

An ***integer*** is an ensemble of identical marks sharing a space.

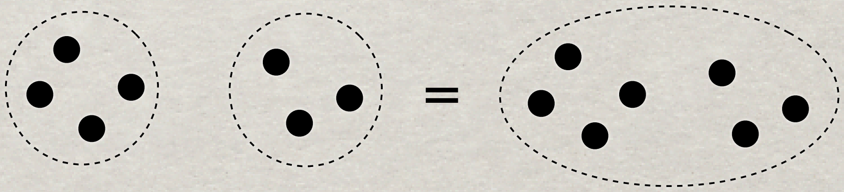
A ***sum*** converts different spaces into the same space.

a + b



Addition is ensembles
sharing a space.

Example: 4 + 3 = 7



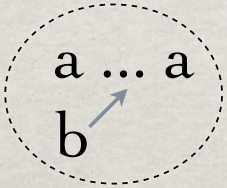
UNIT MULTIPLICATION

A ***product*** converts individual units into ensembles.

• is the unit

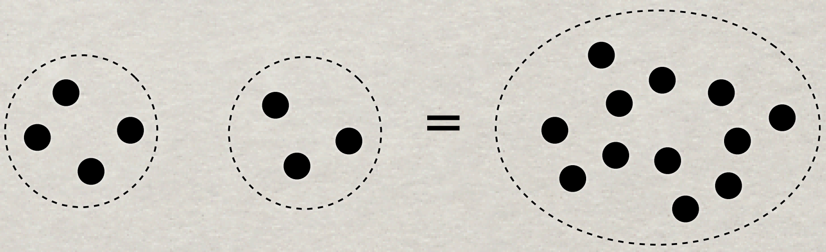
`[substitute a for • in b]` is abbreviated as `[a • b]`

`a × b`



Multiplication is substitution
of ensembles for units.

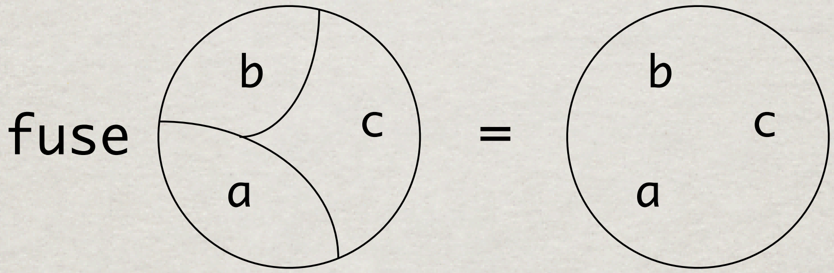
Example: `4 × 3 = 12`



ADDITION AXIOM

ADDITION BY FUSION: to add, remove spatial partitions

fusion is part of mereology



Notation: a|b|c = a b c

Absent group properties:

- zero
- commutativity
- associativity

Arity becomes *concurrent sharing* by many ensembles.

ANNIHILATION AXIOM

−1 ◊ *hole* 0 ◯ *void* 1 ● *whole*

negative one is a
first-class unit

SUBTRACTION BY ANNIHILATION:

to subtract, make whole/hole pairs void

● ◊ =

← void (i.e. nothing)
cannot be represented

Notation: $+a = a_{\bullet}$ $-a = a_{\diamond}$

MULTIPLICATION AXIOMS

MULTIPLICATION BY SUBSTITUTION:

to multiply, replace each unit with an ensemble

Notation: `[substitute a for b in c] = [a b c]`

COMMUTATIVITY OF SUBSTITUTION

`[a b c] = [c b a]`

DISTRIBUTION OF FUSION OVER SUBSTITUTION

`[a|b c d|e] = [a c d] | [a c e] | [b c d] | [b c e]`

Absent group properties:

- zero

Arity becomes *multiple dimensions*.

substitution is a
property of *equality*

COMPARATIVE AXIOMS

GROUP THEORY

$$a + (b + c) = (a + b) + c$$

$$a + b = b + a$$

$$a + 0 = a$$

$$a + (-a) = 0$$

$$a \times (b \times c) = (a \times b) \times c$$

$$a \times b = b \times a$$

$$a \times 1 = a$$

$$a \times (1/a) = 1$$

$$a \times (b + c) = (a \times b) + (a \times c)$$

UNIT-ENSEMBLES

$$a \mid b = a\ b$$

$$[a \bullet b] = [b \bullet a]$$

$$\bullet \ \lozenge \ =$$

$$[a \bullet b \mid c] = [a \bullet b] \mid [a \bullet c]$$

DEPTH-VALUE NOTATION

POSITIONAL NOTATION

Positional notation with a zero place-holder is "one of humankind's greatest achievements".

A uniform base system facilitates simpler algorithms.

1	10	100	1000
... 10^0	10^1	10^2	10^3 ...

Sequential position determines the power of the base.

The same digit can have different meanings

$$3303.3 \quad = \quad 3\times10^3 + 3\times10^2 + 0\times10^1 + 3\times10^0 + 3\times10^{-1}$$

place-holder

POSITIONAL EFFORT

Cognitive and computational load

- Operations require memorization of digit facts
- Carrying is necessary for position overflows
- Algorithms are inherently sequential

Digit facts increase as the base increases

- base-2, 4 facts
- base-10, 100 facts

base-n requires n^A facts
for operators of arity A

Carry overhead increases as the base increases

- base-2, 1/4 addition facts 25%
- base-2, 0/4 multiplication facts 0%
- base-10, 45/100 addition facts 45%
- base-10, 77/100 multiplication facts 77%

facts with
a carry

DEPTH-VALUE NOTATION
(BASE-2)

Standardization converts a unit-ensemble to its minimal form.

–2 ◊ ◊ = (◊)

–1 ◊

0

1 •

2 • • = (•)

3 • • • = (•) •

4 • • • • = (•)(•) = (• •) = ((•))

STANDARDIZATION RULES

• ◊ = annihilate

• • = (•) times 2

(a)(b) = (a b) distribute

DEPTH-VALUE NOTATION

(BASE-10)

n

0	no zero!
1..9	n
10..90	(n)
100..900	((n))

STANDARDIZATION RULES

$a_{\bullet}\ a_{\diamond} =$ annihilate

$10 = (1)$ times 10

$(a)(b) = (a\ b)$ distribute

and 81 (x2) digit facts

3258	(((3)2)5)8	
3258.46	[[(((3)2)5)8]4]6	decimals can be incorporated
3258.46	3(2(5(8[4[6]])))	notation could be inverted

MAXIMAL FACTORED FORM

POLYNOMIAL BASE-10 NUMERAL

$$3258: \quad 3\times10^3 + 2\times10^2 + 5\times10^1 + 8\times10^0$$

MAXIMAL FACTORED BASE-10 NUMERAL

$$10\times(\,10\times(\,10\times(3) + 2) + 5) + 8$$

$$(((3) + 2) + 5) + 8$$ base implicit in boundary

$$(((3)\ 2\ )\ 5\ )\ 8$$ sum implicit in space

VIDEO

Spatial Arithmetic

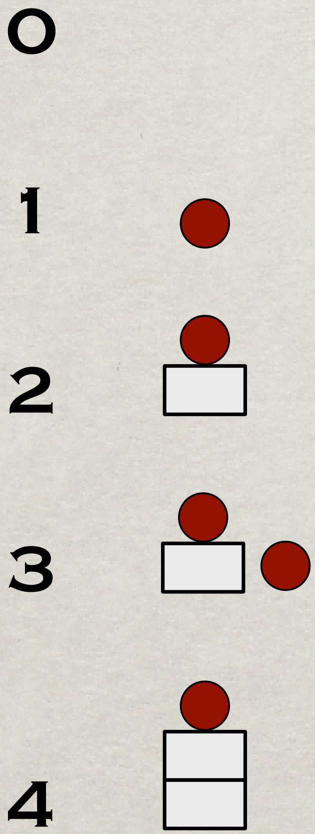
(base-2 enclosures)

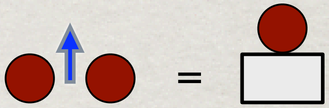
DEMONSTRATION

Spatial Arithmetic

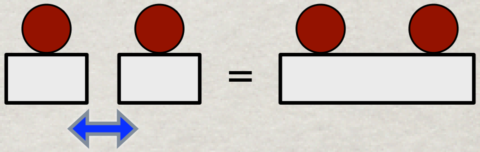
(base-2 blocks)

SPATIAL ARITHMETIC

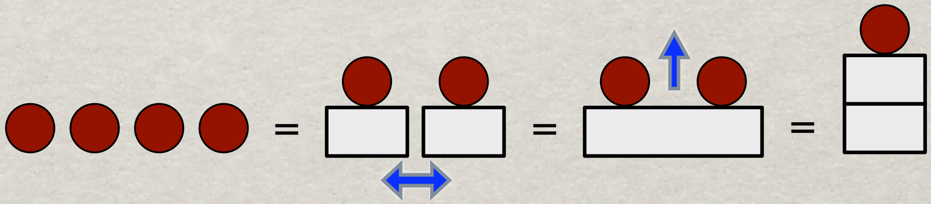




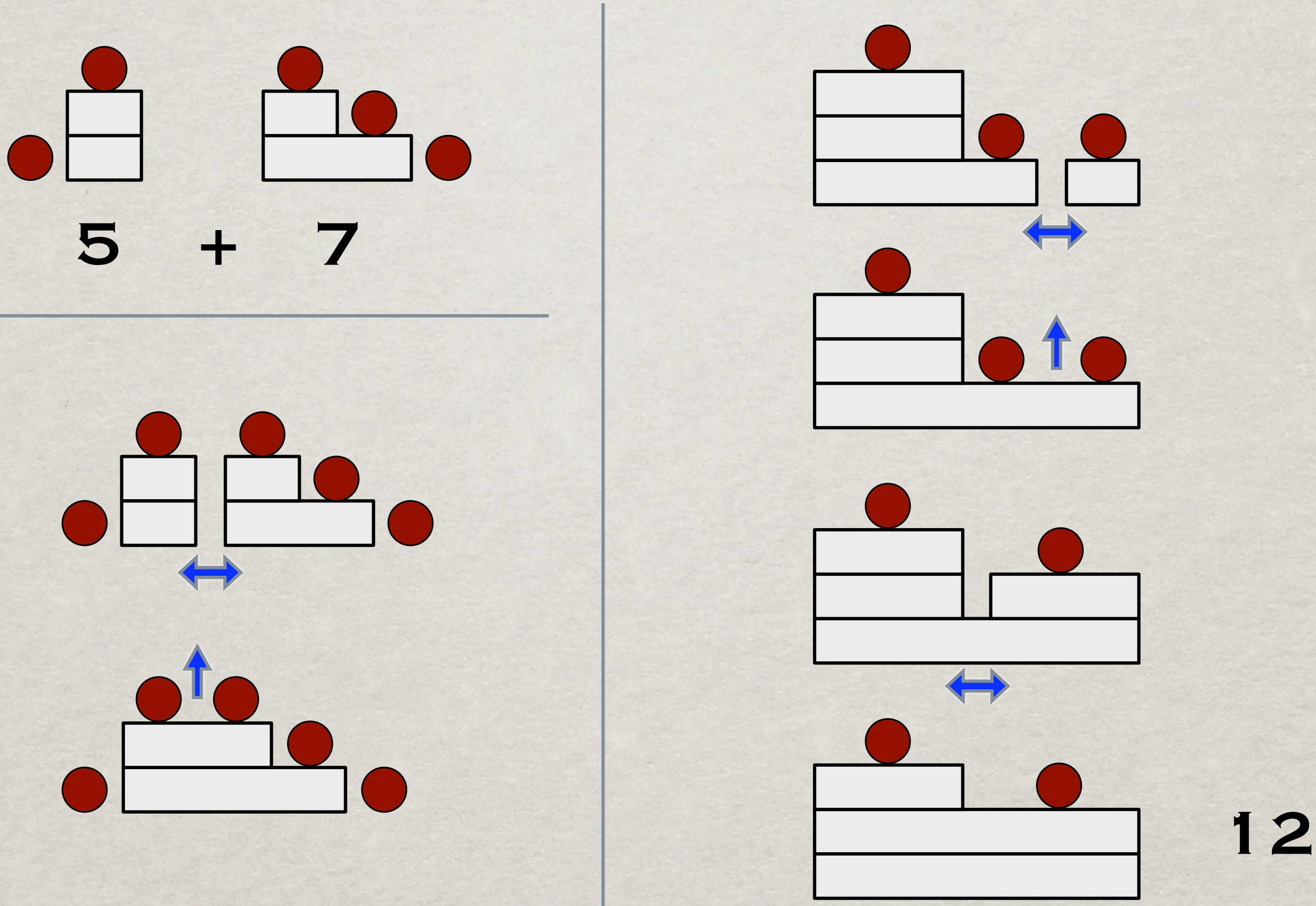
double



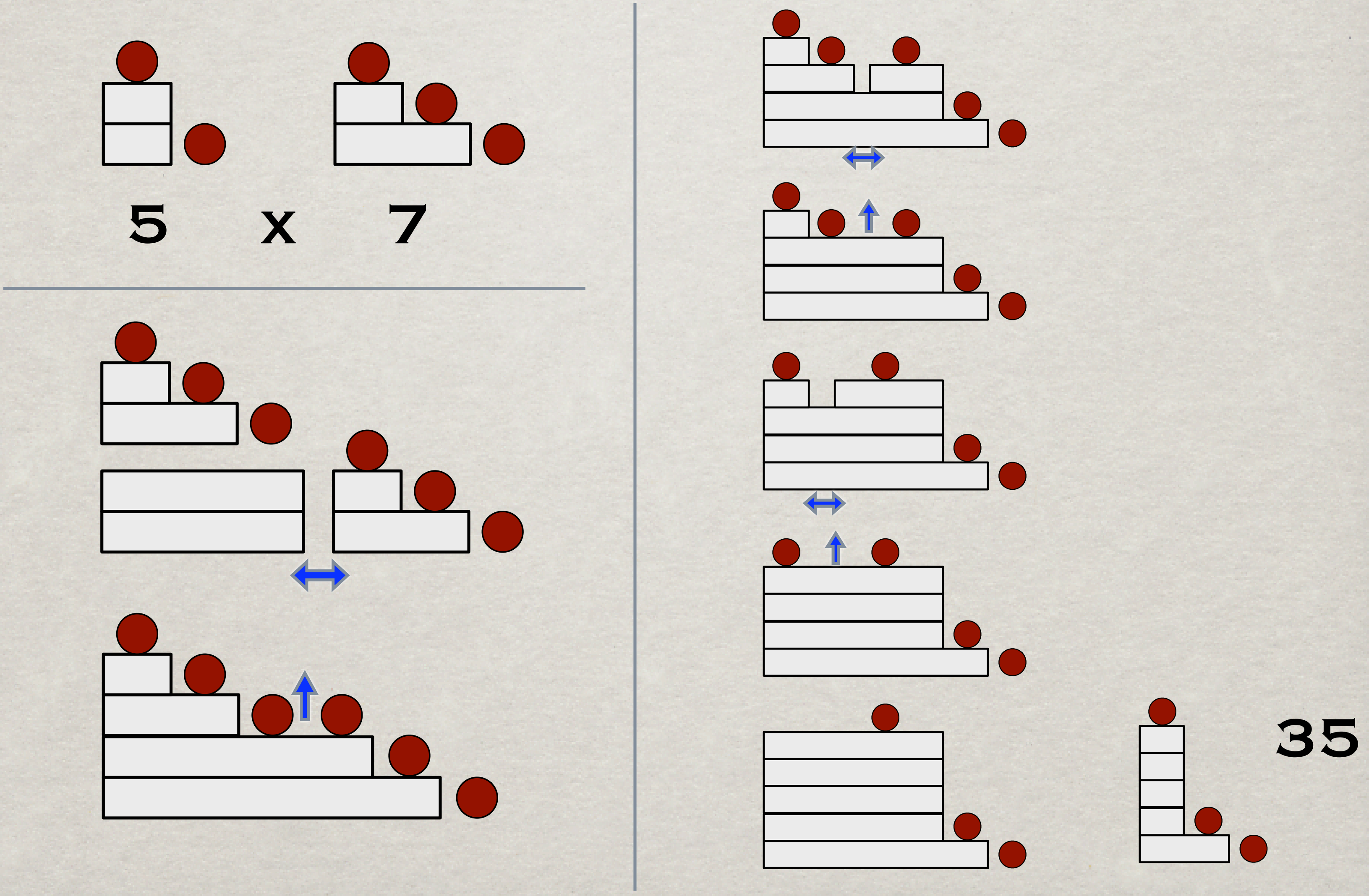
distribute



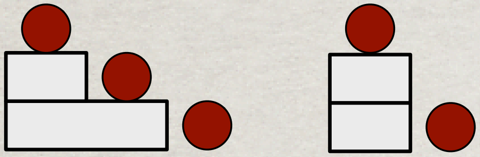
DEMONSTRATION: 5 + 7



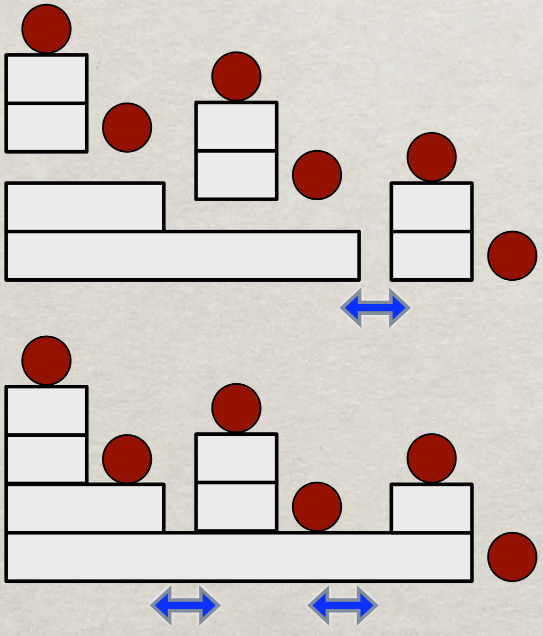
DEMONSTRATION: 5 x 7

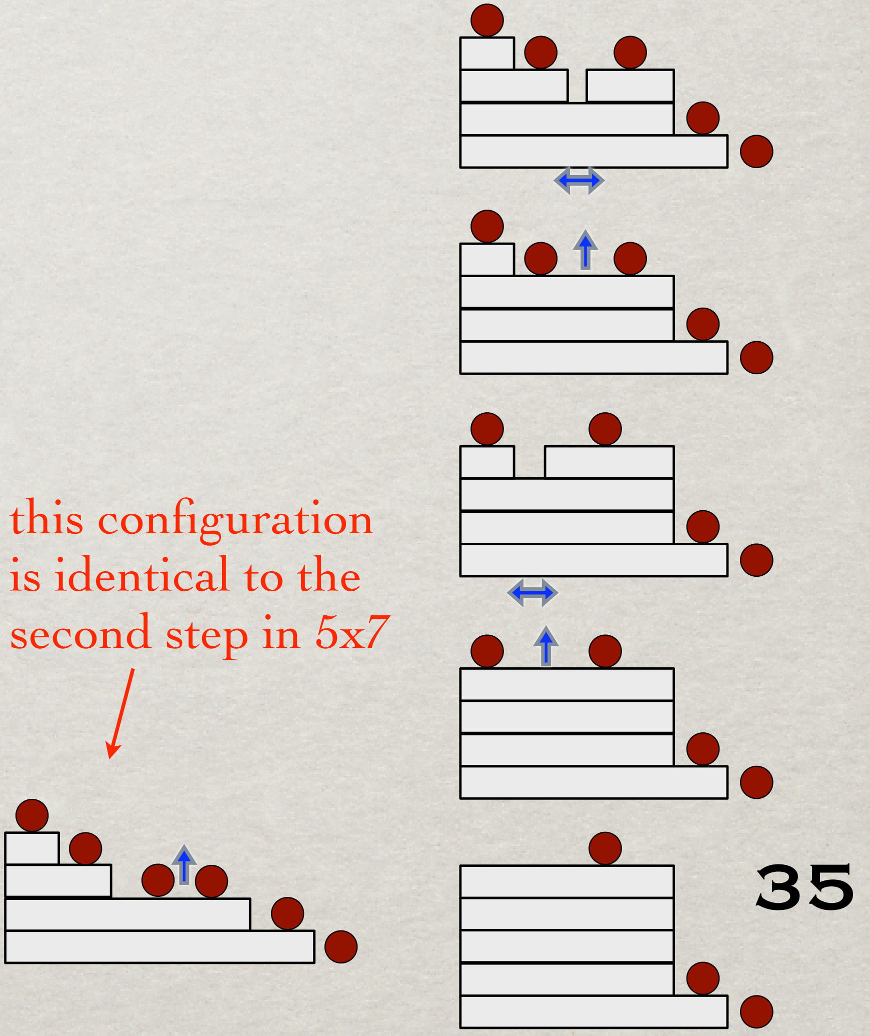


DEMONSTRATION: 7 x 5



7 x 5

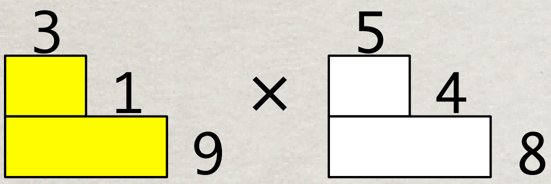


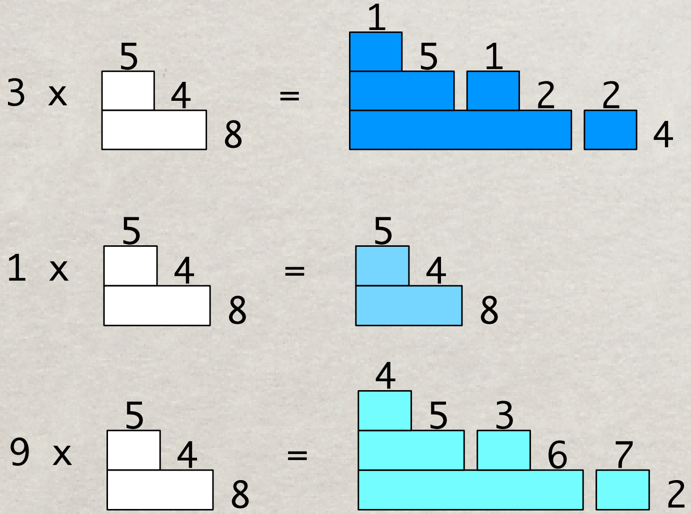


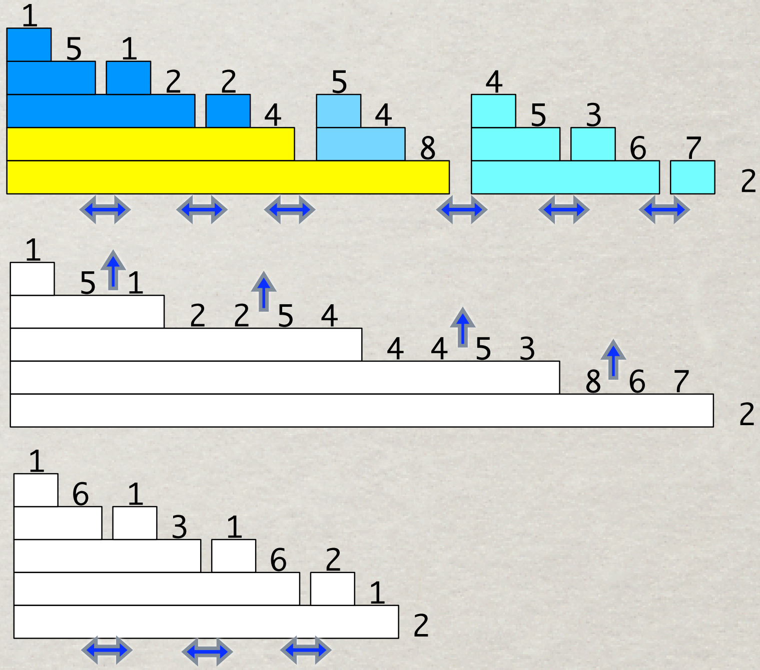
this configuration is identical to the second step in 5x7

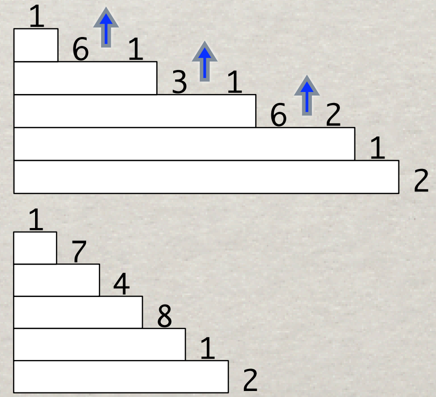
35

BLOCK MULTIPLY (BASE-10)









548
× 319
4932
548
+ 1644
174812

STRUCTURAL QUALITY

	STRUCTURE			
PURPOSE	unit ensembles	Roman numerals	token strings	spatial boundaries
reading/writing	D	C	A	B
computing	C	D	B	A
understanding	A	D	C	B
Grade-points:	7	4	9	10

SUMMARY

The representation of an abstract concept matters, to both humans and machines.

Mathematical meaning can be expressed in formal structures other than strings of meaningless tokens.

Spatial mathematics is rigorous while still respecting the needs of learners.

- historically grounded
- visual, tactile and experiential
- simpler than token-strings
- less cognitive effort
- more humane

THANK YOU!

Comments and suggestions are greatly appreciated.

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This presentation is available in the conference speaker notes,
and on the web at
http://www.wbricken.com/htmls/03words/0303ed/0303-ed.html

SUPPLEMENTAL SLIDES

MORE THAN STRINGS

Our ***delivery media*** for formal ideas are impoverished.

Mathematical structure is richer than token-strings

- diagrams, graphs, maps, paths
- physical and virtual manipulatives
- physical and abstract models
- simulated and actual experiences

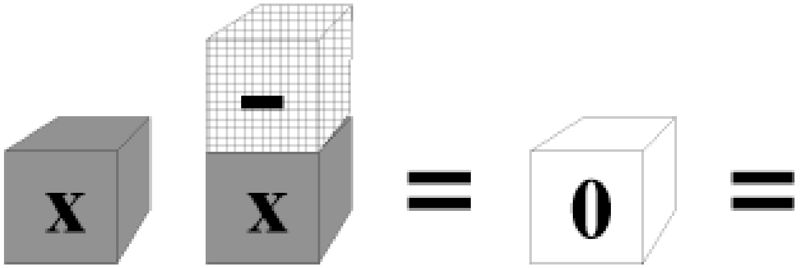
Formal structure can (and should) incorporate human needs

- intuition
- visualization
- physical interaction
- cognitive effort
- comprehension

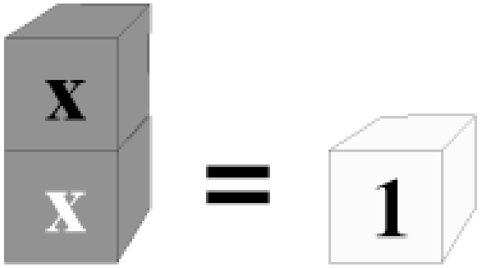
SPATIAL ALGEBRA

INVERSES

many design choices



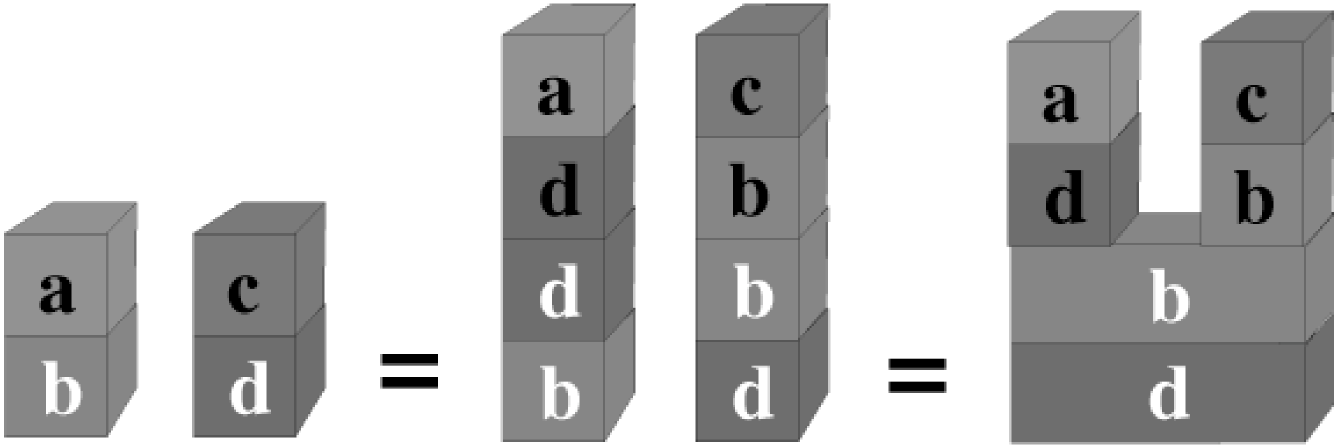
negative blocks CANCEL positive blocks



TOUCHING inverse blocks form the unit

SPATIAL ALGEBRA

FRACTIONS



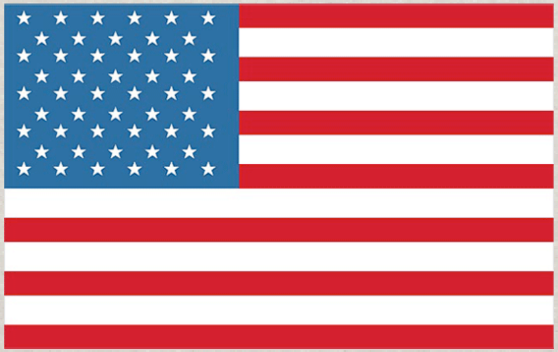
to add fractions:
CONSTRUCT blocks to be joined,
JOIN inverse blocks

ENSEMBLES ON THE FLAG

Fifty stars ⟶ fifty states

Thirteen stripes ⟶ thirteen colonies

- no particular star maps to a particular state
- no particular stripe maps to a particular colony
- spatial arrangement is arbitrary
- color has no meaning
- one-to-one, cardinal but not ordinal



SUBSTITUTION FORMS

Multiplication	$a \times b = b \times a$	$[b \bullet a] = [a \bullet b]$
Division, fraction	b/a	$[b\ a \bullet] = [\bullet\ a\ b]$
Reciprocal	$1/a$	$[\bullet\ a \bullet]$
Exponent	a^2	$[a \bullet a]$

Proof of the multiplicative inverse $a \times (1/a) = 1$

$$[a \bullet [\bullet\ a \bullet]] = [[a \bullet \bullet]\ a \bullet] = [a\ a \bullet] = \bullet$$

$[a \bullet \bullet] = a$

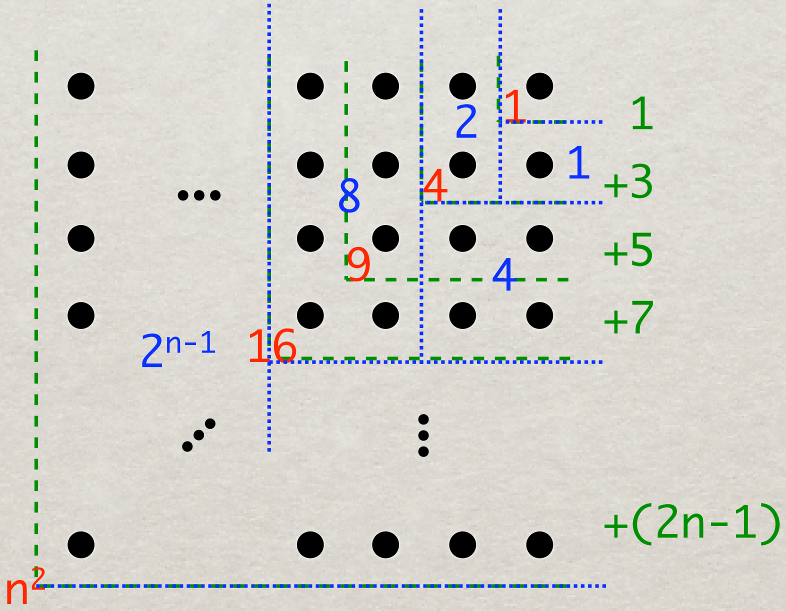
$[\bullet\ a\ a] = \bullet$

super-associativity of substitution

UNIT-ENSEMBLE PROOF

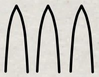
Spatial arrangement of units can provide ***abstract proof***.

$$\sum_{1}^{n}(2i-1) = (\sum_{1}^{n}2^{i-1}) + 1 = n^2$$



NAMED GROUPS

Naming ensembles facilitates counting.

Sumerian cuneiform	3 = 	10 = 
Egyptian hieroglyphics	3 = 	10 = 
Roman numerals	3 = III	10 = X

IIIII = V   VV = X   XXXXX = L   LL = C

Many early number systems included:

- special names for some ensembles
- base-10
- consistent base

They lacked a positional notation with zero place-holders.