How Quantum Logic Differs from Classical Logic: Following Distribution through Two Interpretations of Boundary Mathematics

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1.   Introduction

The entanglement concept of quantum mechanics manifests in mathematical structures as loss of the distribution axiom. The archetypical example used to illustrate this phenomenon is the transmission of light through polarized sheets situated at differing angles from one another. In a simplistic (mathematical) way, the loss of distributivity is expressed as a change from classical logic (CL) to quantum logic (QL).

We follow that change within a system of axioms common to both quantum and classical logic. The system is expressed in a spatial language called Boundary Mathematics (BM). We know the map from boundary logic (BL) to CL. The map from BL to QL is explored here by a map from BL to the behavior of polarized sheets, which is illustrative of QL. By expressing both CL and QL in a common mathematical language BL, each becomes an interpretation of a more fundamental form. The small differences in the axiomatic basis of Boundary Classical Logic (BL) and Boundary Quantum Logic (BQL) identify how the logics differ.

The Losp Engine partially implements this reformulation of the loss of distributivity in the behavior of different Pervasion functions.

First, we characterize distributivity in several notations, and describe simpler versions of the distributive behavior of functions. Then we prove that distributivity is not fundamental, but rather a theorem resulting from the composition of operations in a simpler representation of the basis of CL. These tools will later contribute to an understanding QL.

To develop BQL, we construct a string-based language in which stacking labels in representational space represents the stacking of polarized sheets in physical space. Rotation in space, the other operator in the language, maps cyclic permutation onto physical rotation of the plane of polarization of each sheet around a common axis.

We then develop the transformation axioms of the string language somewhat empirically, by describing the behavior of polarized sheets in the language + transformations, BQL. The identical language is then reinterpreted (by sight) as BL with limits on the generalization of the Pervasion axiom. This permits the known behavior of Pervasion to be applied to QL.

1.1   Distribution

In CL, distribution is often taken as axiomatic. It is, however, a theorem which can be constructed from more elementary concepts. It is common to express the concept of distribution in CL as a rearrangement rule between two binary operators, as in the distribution of AND over OR:

    a and (b or c) = (a and b) or (a and c)

Structurally, the object-operator string "a and" is replicated and attached to each object in the OR relationship. Expressed in functional notation, the structure is:

    f[a, g[b,c]] = g[f[a,b], f[a,c]]

A simpler type of distribution involves a unary function:

    f[g[a,b]] = g[f[a], f[b]]

In CL however, the unary operator NOT does not distribute:

    not(a or b) ≠ (not a) or (not b)

There are unary operators in CL which do distribute, for example the function "ANDTRUE" acts as a "+0" in CL:

    andTrue(a or b) = (andTrue a) or (andTrue b)

        where andTrue(x) = (x and True) = x

In CL, AND distributes over OR, and symmetrically OR distributes over AND. Distribution, however, is not necessarily bi-directional. With integers, * distributes over +, but + does not distribute over *:

    a * (b + c) = (a * b) + (a * c)
    a + (b * c) ≠ (a + b) * (a + c)

From these observations, it is apparent that distribution is not a simple property. BM provides simpler representational structures which support the deconstruction of distribution into its component operations.

1.2   Boundary Distribution

BL provides the operators SPACE and BOUNDARY. The axioms of BL are

    A (   ) = (   )   Dominance
     ((A))   = A   Involution
    A (A B) = A (B)   Pervasion

where capital letters represent arbitrary boundary structures, including void. SPACE is not ordered or grouped, and can contain any number of structures, including none (variarity).

SPACE is associated with the OR operator in CL and BOUNDARY with NOT. The CL law of distribution is expressed as SPACE distributing over DOUBLE-BOUNDARY in BL:

```
A ((B)(C))  =  ((A B)(A C))
```

In BL, BOUNDARY does not distribute over SPACE:

```
(A B)  ≠  (A) (B)
```

Logical distribution is a theorem derived from two more elementary axioms

```
((a    b                )(a         c  ))
((a    b                )(a (      (c)) ))          +involution
((a    b                )(a ((a b)(c)) ))          +pervasion
((a    b                )(a ((  b)(c)) ))          -pervasion
((a ((b)              ))(a ((  b)(c)) ))          +involution
((a ((b)(a ((b)(c)))))))(a ((  b)(c)) ))          +pervasion
((a ((b)(  (  (c)))))))(a ((  b)(c)) ))          -pervasion
((a ((b)(       c  )))(a ((  b)(c)) ))          -involution
((a ((b)(       c  )))                )          -pervasion
  a ((b)(       c  ))                            -involution
```

The + and - signs refer to the direction of rule application, respectively to add or subtract structure.

Pervasion describes the semi-permeability of boundaries:

```
a (b)  =  a (a b)

f[ a, g[b] ]  =  f[ a, g[ f[a,b] ] ]
```

A shorter proof again emphasizes the role that pervasion plays in the definition of distribution:

```
a                       ((  b)(  c))
a                       ((a b)(a c))        +pervasion
((a)                  ) ((a b)(a c))        +involution
((a) ((    a b)(    a c))) ((a b)(a c))        +pervasion
((a) (((a) a b)((a) a c))) ((a b)(a c))        +pervasion
((a) ((( ) a b)(( ) a c))) ((a b)(a c))        -pervasion
((a) (                )) ((a b)(a c))        -occlusion
                        ((a b)(a c))        -occlusion
```

Next we develop QL from observations of the behavior of polarizing sheets, with the intent of comparing BQL to BL.

2.   Polarized Sheets

2.1   Sheets and Stacks

A beam of light strikes the left-most labeled sheet in a stack and passes sequentially through each labeled sheet. Left-to-right sequences of labels are stacks.

Let different sheets be labeled by small letters {a, b, c,...}

Let stacks be labeled by capitals {A, B, C,...}

Eg: A = abcde   stack of five sheets
        B = a   singleton stack
        C ≠   empty stacks are not labeled (do not exist).

The value of a stack is the light it transmits. The central property effecting transmission is the adjacency relationships between sheets.

2.2   Rotations

Sheets have an orientation associated with them. Sheets and stacks can both be rotated about the common central axis; this rotation can be identified with the sheet's orientation relative to a fixed global orientation. For simplicity, we quantize rotations into four 45 degree divisions of a half circle. The half-circle absorbs the symmetry of polarized sheets at 180 degree rotations.

Label the canonical orientations

        u   v   (u)   (v)

to reflect

        0   45   90   135

degree rotations (and their +180n duplicates) of sheet orientation from global zero, which we will call Horizontal. .

        (A)   represents the rotation of A through 90 degrees.

        ((A)) represents a 180 degree rotation of A

Adopt the convention that

        U   is either   u or (u)
        V   is either   v or (v)

2.3   Rules of Rotation

Any stack rotated 180 degrees behaves the same with regard to Horizontal.

        ((A)) = A   **symmetric rotation**

Rotation distributes over space.

        (A B) = (A)(B)   **distributed rotation**

2.4   Value

Rotations and Stacks combine to form the expressions of the language, and the permitted stacks of polarized sheets in the interpretation.

        Eg: A = a ((b) c) (((d))) (e f)   stack prior to rotation
               B = (a b c)   stack rotation
               C = ( )   empty rotation

The value of an expression is the state of light transmitted through a stack. For simplicity, light transmits only in three states {0, .7, 1} for all stacks, corresponding to { blockage, partial-transmission, complete (polarized) transmission }. [The actual transmission of light through a uv configuration is sin 45 degrees, which is approximately .7]

        a   ==>   1   the single sheet transmits completely
        ( ) ==>   0   the empty rotation blocks completely
         UV ==>   .7   45 degree partial transmission

What is surprising in the behavior of polarized sheets is that local changes in adjacency relationships create global changes in the behavior of the stack. Several illustrations of local change follow, the first one is classic:

        1) Placing a v sheet between two orthogonal sheets, u(u), changes the value of the stack from 0 to .7.
        2) Rotating one sheet in the middle of a larger stack can change the value of the stack.
        3) Swapping the location of two sheets can change the value.
        4) Duplicating and distributing sheets can change the value.

The next sections describe the algebraic rules which permit the calculation of the value of a stack. The rules of rotation, for instance, reduce all rotations to one of the canonical forms which describe the permitted sheet orientations. Our focus is on the interaction between adjacent canonical sheets in the context of other sheets.

2.5   Simple Stacks

A non-existent stack is (non)represented as a concatenation of no labels.

Standardize the orientation of the first sheet of every stack to be u. This can be achieved by applying rotation to the entire stack (i.e. rotating the global perspective), or by attaching the global coordinate system to the first sheet. All valid stacks then begin with "u". This makes the global orientation of each stack irrelevant since it is confounded with the first sheet. From another perspective, we could say that global rotation of the stack does not effect light transmission, all that matters is the relative rotation of the sheets in the stack. Admissible stacks now include

```
0-stacks    =
1-stacks    =    u
2-stacks    =    uu  uv  u(u)  u(v)
3-stacks    =    uvu, uvv, uv(u), uv(v), u(v)u, u(v)v, u(v)(u), u(v)(v)
...
```

We can evaluate the relative orientation of two adjacent sheets, generating the three axioms of light transmission.

```
uu    =    1              Duplication
u(u)  =    0              Blocking
uv    =      .7           Partial  Transmission
u(v)  =      .7           Partial  Transmission
```

In Duplication, identically oriented sheets act as one. In Blocking, a sheet and its 90 degree rotation block light totally. Combinations of two different orientations generate Partial Transmission.

2.6   Rules of Light Behavior

For simplicity, sheets impose no filter on light other than polarizing.

Atomic Duplication

$a\ a = a$ The value of duplicate sheets is the value of the sheet.

This does not generalize to stacks, A A ≠ A
Counter example: A = uv(u)

Atomic Blocking

$a\ (a) = (a)\ a = (\ )$ Adjacent orthogonal sheets block.

This does not generalize to stacks, A (A) ≠ ( )
Counter example: A = uv

Contextual blocking

        A ( ) = ( ) A = ( )   Blocked light is final.

Atomic blocking forms a molecular unit, ( ), which is utterly blocking.
Blockage is represented by the empty rotation.

Each of these rules expresses an adjacency relation, but the ordering of the relation (left-right or right-left) is irrelevant. In the notation, small letters express context free operations. The rules apply regardless of the sheets in front or in back of the interacting pair. The rules are of course violated when the adjacency of the pair is broken. Thus these molecular behaviors are conceptually spatial, occurring in parallel and independently.

In an implementation, these rules depend only on pattern identification in the bit stream.

Much of the simplicity of light behavior rests on blockage, ( ). The effect of atomic blocking is total, in that it determines the value of the stack regardless of rotation, arrangement, or even presence of other sheets. Any stack transformation which destroys the adjacency relation between blocking sheets is invalid. Blocking is also spatial; due to its global effects, it is independent and parallel.

In order to encode the universality of the dominance rule,

        A ( ) = ( ) A = ( )

we violate distribution of rotation over space:

        (A)( ) ≠ (A)

In other words, blockage immediately renders rotation irrelevant.

        (A B) = (A)(B)   A, B not empty

The reason that A and B in the above equation must not be empty is simple: the empty rotation does not rotate a stack at all, it rotates the global coordinate system. That is, stacks and globals are incommensurate.

2.7   Simpler Stacks

We can apply the five rules for rotation and light behavior to simplify the range of expressions.

        ((A)) = A   180 degree symmetry
        (A B) = (A)(B)   distribution of rotation over stacks
        a a = a   atomic duplication
        a (*a*) = (*a*) *a* = ( )   atomic blocking
        A ( ) = ( ) A = ( )   contextual blocking

Admissible simplified stacks are now:

```
1-stacks  =  { ( ),  u }
2-stacks  =  { uv, u(v) }
3-stacks  =  { uvu, uv(u), u(v)u, u(v)(u) }
...
```

In UV-notation, stacks that transmit partially have the form uVUVUVU....

```
1-stacks  =  { ( ),  u }
2-stacks  =  { uV }
3-stacks  =  { uVU }
...
```

Value can now be associated with three simple sets:

```
1   =  { u }
 .7 =  {uV, uVU, uVUV, ...}
0   =  { ( ) }
```

In Kleene notation, a stack is either ( ) or it is uVU*[V].

All non-blocked stacks greater than two sheets repeat the pattern VU. The value of a stack does not depend upon the number of occurrences of VU, so we can arbitrarily choose zero occurrences as canonical. Then stacks of interest map one-to-one on values:

```
STACKS  =  { ( )=0,  uV=.7,  u=1 }
```

And since the initial orientation is arbitrary, we can let it be U rather than u, yielding

```
STACKS  =  { ( ), U, UV }
```

2.8   State Transition Diagram

Another way of expressing the relation between stacks and their value is a state transition diagram, in which the state of light is always one of { 1, .7v, .7u, 0 } and state changes when moving one sheet to the right in a stack.
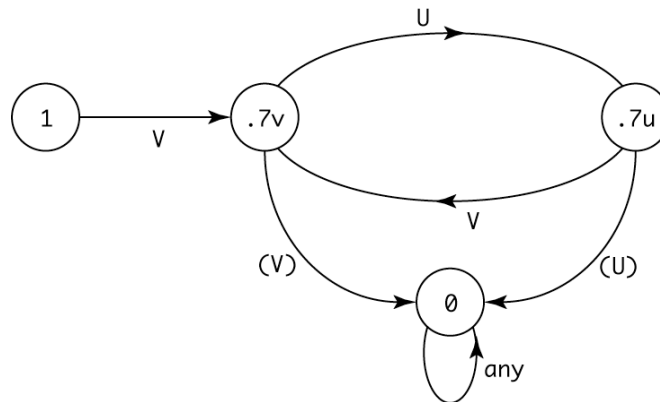
3.   Discussion

3.1   Distribution and Rotation

Consider the CL distributive rule interpreted as rotations:

$$a \; ((b)(c)) = a \; ((b))((c)) = a \; b \; c$$

$$((a \; b)(a \; c)) = ((a \; b))((a \; c)) = ((a))((b))((a))((c)) = a \; b \; a \; c$$

This provides a very elementary form of distribution, single operator distribution:

$$f[\; a, \; f[b,c] \;] = f[\; f[a,b], \; f[a,c] \;]$$

Compare to standard distribution:

$$f[\; a, \; g[b,c] \;] = g[\; f[a,b], \; f[a,c] \;]$$

In CL, single operator distribution holds for OR:

$$a \; or \; (b \; or \; c) = (a \; or \; b) \; or \; (a \; or \; c)$$

In BL, this is expressed as

$$a \; b \; c = a \; b \; a \; c$$

In QL, single operator distribution does not hold for adjacency. The counter example is: a is u, b is v, c is (u)

$$uv(u) \neq uvu(u)$$

uv(u) is the stack with the surprising polarized behavior. Our surprise is due to the expected distribution not working. But what is happening is that we are violating the adjacency structure of the stack, in particular, we are deconstructing a blocking unit a(a). Note that blockage does permit distribution over space as a special case:

$$(\; \;) \; b \; c = (\; \;) \; b \; (\; \;) \; c$$

This is not distribution at work, rather it is a consequence of Dominance.

3.2   Adjacency

The adjacency relations in a stack determine the value of the stack. Any stack operation which modifies the adjacency relations can change the value of the stack. Adjacency is a particularly simple way to express the context dependency (or location dependency) of transmission rules.

The simplest case of context or adjacency dependency is inserting a sheet between two sheets that form a molecule. For example:

uu ≠ uvu

u(u) ≠ uv(u)

Naturally the stack value can be changed by rotating sheets. This change can be global, even though the rotation is local.

    u(v)u ≠ u(u)u   (v) --> (u)

    u(u)v ≠ uvv   (u) --> v

    u(u)vuvu ≠ uvvuvu   (u) --> v

The concept of commutativity can be expressed by changing the location of sheets, or by selected rotations that are equivalent to changing location:

    uv(u)(v) ≠ u(u)v(v)   v --> (u)   (u) --> v

The concept of associativity presupposes an ordering to the application of adjacency. It is naturally irrelevant since molecular effects are spatial and not linear. If we use square brackets to represent ordering of adjacency rules, associativity becomes:

    [uv](v) = u[v(v)]

Molecular adjacency dominates associativity in that the value of [uv] is less relevant to the value of the stack than is the relationship between adjacent sheets, here [v(v)].

The above observations make it clear that distributivity over adjacency cannot hold. In fact, distributivity and adjacency are conceptual opposites.

3.3   Theorems

    (( )) = ( )   Utter Dominance
    A B = ((A)(B))   Non-duality
    a (a) = a ( )   Atomic Pervasion

Utter dominance means that once blocking occurs, rotation becomes irrelevant. Rotating a blocked stack still leaves it blocked. This theorem can be seen as defining an essentially imaginary value, as in the Varela boundary calculus. It is the initial rotation of the empty stack that is imaginary (i.e., outside of the model of stacks).

Non-duality makes it clear why single operator distribution occurs in BQL: there are simply not two binary operators to form a lattice around.

Very limited pervasion is , of course, the heart of the change from CL to QL. In CL, Duplication and Pervasion can be shown to be the same rule, so that Atomic Pervasion and Atomic Duplication are just symmetrical endpoints (complete blockage vs. no change in transmission) of the same process. What is common to the symmetrical endpoints is that they both require two references to a sheet's orientation.

3.4   Rule Summary

```
((A)) = A                           A is any stack
((  )) = (   )
A (  ) = (   ) A = (   )             A is any stack
(A B) = (A)(B)                       A, B  are not blocked stacks
a a = a
a (a) = (a) a = (   )
```

The two rules

```
((A)) = A                           Involution
A (  ) = (   ) A = (   )             Dominance
```

are the same for CL and for QL. Note that both are independent of adjacency. The two rules

```
a a = a                             Atomic   Duplication
a (a) = (a) a = (   )                Atomic   Pervasion
```

are the same for both CL and QL, with the exception that both rules are atomic and without generalization in QL. The failure to generalize rests upon the requirement of atomic adjacency. Note that the spatiality of BL can be degraded to adjacency without problem. We merely need to add associativity rules. And in fact, CL notation assumes linearity and thus adjacency, although associative rules undermine atomic adjacency.

The loss of generality of Pervasion means the loss of distribution of SPACE over DOUBLE-BOUNDARY, since distribution as a theorem relies heavily on generalized Pervasion. However, distribution is gained in rotation (SINGLE-BOUNDARY) over SPACE. That is, this interpretation of QL is still distributive, but different operators distribute.

```
(A B) = (A)(B)                      Distribute rotation over space
```

The difference between CL and QL is now localized. In CL, the unary operator (boundary interpreted as NOT) does not distribute over the binary operators (space as OR, and double-boundary as AND). But the two binary operators mutually distribute. In QL, the unary operator (boundary interpreted as ROTATE) distributes over the binary (space as adjacency), while the dual binary operator (double-boundary as 180 degree rotation) is trivialized by the unary distribution, in that the two binary operators collapse to being equal.

CL: f = OR   g = AND   h = NOT

```
f[ a, g[b,c] ] = g[ f[a,b], f[a,c] ]
g[ a, f[b,c] ] = f[ g[a,b], g[a,c] ]
```

QL: f = ADJACENT   g = 180-ROTATE   h = 90-ROTATE

```
h[f[a,b]] = f[ h[a], h[b] ]
```

One might say that CL exhibits binary duality and distribution at the cost of unary distribution. QL exhibits unary distribution at the cost of binary distinctions.

3.5   Deconstructed Distribution

In BL, duality is deconstructed as double-boundary. From a boundary perspective on CL, distribution is more complex

BL: f = SPACE   g = DOUBLE-BOUNDARY   h = BOUNDARY

g[a,b] = h[f[h[a], h[b]]]

f[a, h[f[h[b], h[c]]]] = h[f[h[f[a,b]], h[f[a,c]]]]

This form simplifies by making a composite operator hf, BOUND-SPACE.

f[a, hf[h[b], h[c]]] = hf[hf[a,b], hf[a,c]]

Since space is within every boundary, the h operator can be seen to be a the binary operator hf with one argument void:

f[a, hf[hf[,b], hf[, c]]] = hf[hf[a,b], hf[a,c]]

Finally, the f operator can be converted to hf by enclosing the space in a double boundary. The rule ((A)) = A assures no change of value.

hf[, hf[a, hf[hf[,b], hf[, c]]]] = hf[hf[a, b], hf[a, c]]

From this perspective, we can see that two operator distribution can be expressed as single operator distribution. Since all operators are the same, for clarity we can omit the hf prefix in front of each bracket:

[, [a, [[, b], [, c]]]] = [[a, b], [a, c]]

Now distribution is simply a form of void-substitution across double-function application, In the above, we have exchanged a for void across deeper double boundaries.

Functional notation is greatly simplified by letting space and boundary represent themselves in infix notation:

((a ((b)(c)))) = ((a b)(a c))

The double-boundary immediately identifies the complexity of distribution as a composite operation.

3.6 Imaginaries

Imaginaries occur directly in the construction of the empty rotation:

$$((\ )) = (\ ) \qquad\qquad \textbf{Imaginary \ \ Rotation}$$

This is a form of Atomic Imaginary that does not generalize, thus it is related to Varela's imaginary state. Varela's arithmetic initials (which is morphic to Kleene 3-valued logic) do not apply, since

$$((\ )) \neq$$

Varela's algebra includes distribution and thus also does not apply. Kauffman's four valued logic has the same inapplicability.

Complex numbers occur in the form of the four canonical orientations { u, v, (u), (v) }. We could have developed the notation purely from 45 degree rather than 90 degree rotations, using the atoms

$$\{ \ u, \ (u), \ ((u)), \ (((u))) \ \}$$

where v = (u) and ((u))u blocks. This approach seems to add nothing of interest. However the following map connects imaginaries to rotations:

```
u = 1        (u) = -1
v = i        (v) = -i

 ( )     =  0
space    =  +
boundary =  -
```

Here transmission has the value 1; blocking, the value 0. uv-partial transmission is 1 + i, which has the magnitude .7 constructed from the norm of the complex number. This is quite analogous to evaluation of collapsing probability waves.

4. Conclusion

The highlights of this discussion follow:

1. Distribution is not a fundamental property of logic; Pervasion is.

2. The same mathematical mechanism underlies both CL and QL (as exemplified by the behavior of polarized sheets).

3. The single important difference between CL and QL is that the atomic adjacency rules of QL do not generalize, thus breaking all concepts of rearrangement of elements, including associativity, commutativity, and distribution. These are regained when atomic blocking is molecular and therefore not deconstructable.

5. References

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