The Mathematics of Boundaries: A Beginning

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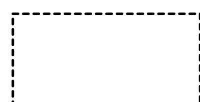
Abstract. The intuitive properties of configurations of planar non-overlapping closed curves (boundaries) are presented as a pure boundary mathematics. The mathematics, which is not incorporated in any existing formalism, is constructed from first principles, that is, from empty space. When formulated as pattern-equations, boundary algebras map to elementary logic and to integer arithmetic.

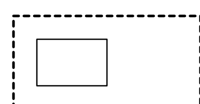
1 *De Novo* Tutorial

Boundary mathematics is a formal diagrammatic system of configurations of non-overlapping closed curves, or *boundary forms*. Transformations are specified by algebraic equations that define equivalence classes over forms. As spatial objects, forms can be considered to be *patterns*, and transformations to be *pattern-equations* that identify valid pattern substitutions. The algebra of boundaries is novel, and is not incorporated within existing mathematical systems. Boundary mathematics provides a unique opportunity to observe formal structure rising out of literally nothing, without recourse to, or preconceptions from, the existing logical, set theoretic, numeric, relational, geometric, topological, or categoric formal systems that define modern mathematics. The strategy is to build a *minimal* diagrammatic language and an algebra for that language, using substitution and replacement of equals as the only mechanism of computation.

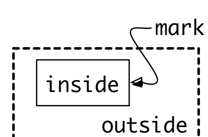
1.1 Language



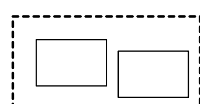
I. Set aside a space to support drawing. Notice that it is framed.



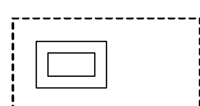
II. Notice that the frame indicates an identifiable (i.e. framed) empty space, S. Following a minimalist strategy, draw the only observable thing (the frame) inside the only available space (S).



III. Call the representation of the frame a *mark*. Notice that S has changed from empty to not-empty, and that there are now three identifiable diagrammatic proto-structures: the mark, the inside of the mark, and the outside of the mark.



IV. Notice that replicate marks can now be drawn in two places, on the inside and on the outside of the original mark. Construct each variety. Drawing on the outside is `SHARING;` drawing on the inside is `BOUNDING`.



V. We have constructed a language consisting of three structurally different forms, and one absence of form. This language has two operators for constructing further forms. `SHARING` and `BOUNDING` can be applied indefinitely, each application adding one mark. Four forms can be constructed from 3 marks, nine forms from 4 marks.

D. Barker-Plummer et al. (Eds.): Diagrams 2006, LNAI 4045, pp. 70 – 72, 2006.

Let's represent the mark more succinctly as (). The language of marks maps onto well-formed parenthesis structures without ordering. The dotted frame is part of the metalanguage that permitted description of mark drawings, and is no longer of use.

1.2 Algebra

VI. Let "=" mean *is-structurally-identical-to*: ()=(), ()()=()(), (())=(()). Since marks are constructed in space, without ordering, note that ()(())=(())().

VII. Let "≠" mean *is-not-identical-to*: ()≠()(), ()≠(()), ()()≠(()).

VIII. There are eight possible ways that the three forms can be collected into groups: {}, {()}, {()()}, {(())}, {(),()()}, {(),(())}, {()(),(())}, {(),()(),(())}

IX. Let "=" also mean *in-the-same-collection*. There are four possible new equalities: ()=()(), ()=(()), ()()=(()), ()=()()=(())

X. Discard the universal collection, since it does not distinguish between forms. There are then three possible arithmetics that can be constructed from the remaining three equalities.

Arithmetic I:	()=()(), ()≠(()), ()()≠(())	--	{(),()()}/{(())}
Arithmetic II:	()≠()(), ()=(()), ()()≠(())	--	{(),(())}/{()()}
Arithmetic III:	()≠()(), ()≠(()), ()()=(())	--	{()(),(())}/{()}

2 Interpretations

The minimalist mark-arithmetics map to elementary logic and to integer arithmetic, suggesting a diagrammatic foundation for conventional mathematics. Each mark-arithmetic can be generalized to an algebra by including variables that stand in place of arbitrary forms. Boundary logic is such a generalization [1][2]. Peirce first developed this logic in its implicative form as Entitative Graphs [3, 3.456-552 (1896)]. Spencer Brown [4] presents an algebraic version of *boundary logic arithmetic*, using Mark-Arithmetic I (below). Kauffman constructs *boundary integer arithmetic* from Mark-Arithmetic III [5].

2.1 The Map to Logic

The partitions of Mark-Arithmetic I, {(),()()}/{(())}, assert the *Call* rule ()=()(); idempotency is the primary differentiator between logic and numerics. The mark is TRUE, while forms SHARING a space are interpreted as joined by disjunction. BOUNDING is negation.

Spencer Brown's innovation was to equate (()) with nothing at all, that is, with the contents of the dotted frame prior to drawing the first mark. This created two Boolean equivalence classes while using only one symbol. Truth is confounded with existence, a capability unique to the spatial structure of the mark. The *Cross* rule generalizes to algebra as both *Occlusion*, which terminates proofs, and *Involution,* which enforces depth parity. The workhorse of boundary logic is *Pervasion*, which has no analog in conventional techniques. The curly brace is a *meta-boundary*, standing in place of any spatially intervening content, including none. Curly braces identify *semipermeable boundaries*, a defining characteristic of boundary logic. Boundary logic proof of

f	*void*
t	( )
¬A	(A)
A→B	(A) B
A ∨ B	A B
A ∧ B	((A)(B))

CALL	()() = ()
CROSS	(()) = *void*
OCCLUSION	(A ()) = *void*
INVOLUTION	((A)) = A
PERVASION	A {A B} = A {B}

the *self-distributive law of the conditional* is illustrated to the right. The three reduction rules will reduce any TRUE form to a mark. A logical interpretation of *Pervasion* would delete deeply nested connectives and their arguments that match forms anywhere in their context.

```
(p→(q→r))→((p→q)→(p→r))   theorem
 ((p)(q) r) ((p) q) (p) r  transcribe
 (   (q)  ) (   q) (p) r   pervasion
 (        ) (   q) (p) r   pervasion
(((        ) (   q) (p) r)) involution
(                        )  occlusion
```

2.2 The Map to Integers

The partitions of Mark-Arithmetic III, {()(),(())}/{()}, distinguish mark from the other forms. Here, a mark is represented by a centered dot, • , for visual ease. It is interpreted as the integer unit, 1. Forms SHARING the same space are interpreted as added together, similar to stroke arithmetic used to tally units. Since by construction, ••=(•), BOUNDING is interpreted as doubling the unit, creating a boundary place notation.

```
0        void
1     •
2    •• = (•)
3   ••• = (•)•
4  •••• = ((•))
...
```

The *unit double* rule can be generalized to an algebraic *double* rule. An elegant *merge* rule can be derived, as well, by observing that forms can be partitioned for doubling in two ways, the boundary analog of the rule of distribution. This is illustrated by the structure of the number four:

```
   ••••  = (•) (•) =  (••)  = ((•))
1+1+1+1 = 2*1+2*1 = 2(1+1) = 2*2*1
```

```
UNIT DOUBLE   •• = (•)
DOUBLE        A A = (A)
MERGE     (A)(B) = (A B)
```

Boundary multiplication is achieved by substituting forms for units. To multiply B by A, *substitute* A *for every* • *in* B. Six is highlighted in this example, 5*6:

```
6=((•)•)     5=((•))•     5*6=((((•)•)))((•)•)
```

The above result is immediately equal to 30, no further computation is required. However, the reduction rules can be applied to reach a canonical shortest form of the result. Both 5*6 and 6*5 are illustrated below:

```
30 = ((((•)•))) ((•)•)   5*6       ((((•))•)((•) )•)   6*5
     ((((•)•))    (•)•)  merge     ((((•))•   (•) )•)  merge
     ((((•)•)      •)•)  merge     ((((•)       •)•)•) merge
```

Boundary integers differ from boundary logic in both type of boundary (impermeable vs semipermeable) and in type of multiplication (substitution vs imposed structure).

References

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